Chapter 1

Vector Algebra

1.1 Rectangular Coordinates in 1-Space, 2-Space and 3-Space

The *Euclidean Geometry* is the study of geometric elements (points, lines, planes, etc), relations between elements and configurations. The simplest geometric elements and basic relations between them were introduced for the first time in an axiomatic way by Euclid (IV$^{\text{th}}$-III$^{\text{th}}$ centuries B.C.), in his famous book "Elements".

A complete axiomatic system in geometry was given by D.Hilbert (1862-1943) in "The Fundamentals of Geometry" in 1899. Two other axiomatic systems in Euclidean geometry were introduced by G.D.Birkhoff (1884-1944) in 1932, and by H.Weyl (1885-1955) in 1917.

The basic axioms and relations are well-known from the Mathematics in secondary school or high school. They define the Euclidean plane $\mathcal{E}_2$ and the Euclidean space $\mathcal{E}_3$.

The *Analytic Geometry* is the study of geometric configurations by using the coordinates method, which works not only in $\mathcal{E}_2$ or $\mathcal{E}_3$, but also in a n-dimensional Euclidean space.

1.1.1 The Space of Coordinates $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the set of all ordered n-tuples of real numbers, i.e.

$$\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}.$$

Therefore, an element x of $\mathbb{R}^n$ has the form $x = (x_1, \ldots, x_n)$; the real numbers x_1, ..., x_n are called *components* of x. Recall that two n-tuples $(x_1, \ldots, x_n)$

and $(y_1, \ldots, y_n)$ are equal if and only if their components are respectively equal:

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \Longleftrightarrow x_1 = y_1, \ldots, x_n = y_n.$$

1.1.2 Cartesian Coordinates on $\mathcal{E}_1$

Let $\mathcal{E}_1$ be the 1-dimensional Euclidean space, i.e. a line d together with one of the above mentioned system of axioms. Let O and A be two different points, fixed on d, such that $OA = 1$ (see Figure 1.1).

Figure 1.1:

One obtains an orientation on the line d (from O to A) and one can introduce the function

$$f_1 : \mathcal{E}_1 \to \mathbb{R}, \quad f_1(P) = x_P,$$

where $|x_P| = OP$ and $\begin{cases} x_P \geq 0 & \text{if} \quad P \in [OA \\ x_P < 0 & \text{if} \quad P \notin [OA \end{cases}$. Hence, f is bijective and one can associate to any point $P \in \mathcal{E}_1$ a unique real number x_P. One says that Ox is a *Cartesian system of coordinates* on $\mathcal{E}_1$, having the *origin* O and the *axis* Ox, while x_P is said to be the *coordinate* of P. We shall use the notation $P(x_P)$.

1.1.3 Rectangular Coordinates on $\mathcal{E}_2$

Let $\mathcal{E}_2$ be the 2-dimensional Euclidean space, i.e. a plane π together with a system of axioms mentioned at the beginning of this chapter. Let $O \in \mathcal{E}_2$ be a fixed point d and d' be two orthogonal lines, passing through O. One can choose, on each of the lines d and d', a Cartesian system of coordinates, having the same origin O. Suppose they are denoted by Ox respectively Oy.

If P is an arbitrary point of $\mathcal{E}_2$, let x_P and y_P be the coordinates of the orthogonal projections P_x and P_y on Ox respectively Oy. One can associate to any point $P \in \mathcal{E}_2$ a unique pair $(x_P, y_P) \in \mathbb{R}^2$, so that one obtains a bijection

$$f_2 : \mathcal{E}_2 \to \mathbb{R}^2, \quad f_2(P) = (x_P, y_P).$$

The *origin* O together with the *axes* Ox and Oy form a *Cartesian system of coordinates* Oxy *on* $\mathcal{E}_2$, while the *rectangular coordinates* of P in this system are (x_P, y_P) (see Figure 1.2).

Figure 1.2:

The axes d and d' divide the plane $\mathcal{E}_2$ into four domains which are named *quadrants* and are denoted by I, II, III, IV. One can characterize to which quadrant of the plane a point P belongs, by looking at the signs of coordinates (x_P, y_P) of the point P (which is not situated on the coordinates lines Ox, Oy).

Quadrant	I	II	III	IV
x_P	$+$	$-$	$-$	$+$
y_p	$+$	$+$	$-$	$-$

A simple replacement in Pythagora's Theorem shows that the *length* of the segment $[P_1P_2]$, where $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, is given by the formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1.1}$$

If the point P divides the segment $[P_1P_2]$ into the ratio k, i.e. $\dfrac{PP_1}{PP_2} = k$, then the coordinates of P are

$$P\left(\frac{x_1 + kx_2}{1+k}, \frac{y_1 + ky_2}{1+k}\right). \tag{1.2}$$

1.1.4 Rectangular Coordinates in $\mathcal{E}_3$

Let $\mathcal{E}_3$ be the 3-dimensional Euclidean space, O be a fixed point and d, d', d'' be three pairwise orthogonal lines, passing through O. Choose, on the lines d, d' and d'', the Cartesian systems of coordinates Ox, Oy respectively Oz, having the same origin O (see Figure 1.3).

For an arbitrary point in $P \in \mathcal{E}_3$, denote by x_P, y_P and z_P the coordinates of its orthogonal projections P_x, P_y and P_z on d, d' respectively d''. One can define the bijection

$$f_3 : \mathcal{E}_3 \to \mathbb{R}^3, \quad f_3(P) = (x_P, y_P, z_P).$$

Figure 1.3:

One introduced on $\mathcal{E}_3$ the right *rectangular coordinates system* (or the Cartesian system) $Oxyz$, having the following elements:

- the *origin* O;
- the *coordinate lines* (or *axes*) Ox, Oy, Oz;
- the *coordinate planes* Oxy, Oyz, Ozx;

Examples.

1) The origin O has the coordinates $(0, 0, 0)$.

2) The points situated on Ox, Oy and Oz are of coordinates $(x, 0, 0)$, $(0, y, 0)$, respectively $(0, 0, z)$.

3) The points situated on Oxy, Oyz and Ozx have the coordinates $(x, y, 0)$, $(0, y, z)$ respectively $(x, 0, z)$.

The three coordinate planes divide the space $\mathcal{E}_3$ into eight domains, denoted by I, II, ..., VIII. These domains are defined by the signs of the coordinates of the points they contain, as in the following table:

Domain	I	II	III	IV	V	VI	VII	VIII
x_P	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
y_P	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$
z_P	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$

Theorem 1.1.4.1. (the distance formula) *The distance between the points* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *is given by*

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1.3}$$

Proof: Construct a right parallelepiped having P_1 and P_2 as opposite vertices and the faces parallel to the coordinate planes, as in Figure 1.4. One

Figure 1.4:

has

$$P_1A = |y_2 - y_1|,$$

$$AB = |x_2 - x_1|,$$

$$BP_2 = |z_2 - z_1|.$$

Since

$$P_1B^2 = P_1A^2 + AB^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2,$$

then

$$P_1P_2^2 = P_1B^2 + BP_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

and the formula 1.3 is obtained. $\square$

Theorem 1.1.4.2. *If the point P divides the segment $[P_1P_2]$, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, into the ratio k (i.e. $\dfrac{PP_1}{PP_2} = k$), then the coordinates of P are*

$$\left(\frac{x_1 + kx_2}{1+k}, \frac{y_1 + ky_2}{1+k}, \frac{z_1 + kz_2}{1+k}\right). \tag{1.4}$$

Proof: Take $P(x_P, y_P, z_P)$ and construct $PP' || P_1A$, with $P' \in AP_2$ (see Figure 1.4). The similarity of the triangles $\Delta PP'P_2$ and ΔP_1AP_2 yields

$$\frac{PP'}{P_1A} = \frac{PP_2}{P_1P_2} = \frac{1}{k+1},$$

which is equivalent to

$$\frac{|y_2 - y_P|}{|y_2 - y_1|} = \frac{k}{k+1}.$$

Supposing that $y_2 > y_1$ and $y_P > y_1$ (otherwise, the calculations are analogous), one obtains $y_P = \dfrac{y_1 + ky_2}{1+k}$.

Similarly, $x_P = \dfrac{x_1 + kx_2}{1+k}$ and $z_P = \dfrac{z_1 + kz_2}{1+k}$. $\square$

Remark. If M is the midpoint of the segment $[P_1P_2]$, determined by $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, then $k = 1$ and the coordinates of M are

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right). \tag{1.5}$$

1.1.5 Exercises

1. Give the coordinates of the vertices of the rectangular parallelepiped whose sides are the coordinate planes and the planes $x = 1$, $y = 3$, $z = 6$.

2. Describe the locus of points $P(x, y, z)$ in each of the following situations:

   a) $xyz = 0$;
   b) $x^2 + y^2 + z^2 = 0$;
   c) $(x+1)^2 + (y-2)^2 + (z+3)^2 = 0$;
   d) $(x-2)(z-8) = 0$;
   e) $z^2 - 25 = 0$;

3. Show that the given points are collinear:

    a) $P_1(1, 2, 9), P_2(-2, -2, -3), P_3(7, 10, 6)$;

    b) $Q_1(2, 3, 2), Q_2(1, 4, 4), Q_3(5, 0, -4)$.

4. Find x if:

    a) $P_1(x, 2, 3), P_2(2, 1, 1)$ and $P_1P_2 = \sqrt{21}$;

    b) $Q_1(x, x, 1), Q_2(0, 3, 5)$ and $Q_1Q_2 = 5$.

5. The coordinates of the midpoint of the segment $[P_1P_2]$, determined by $P_1(x_1, y_1, z_1)$ and $P_2(2, 3, 6)$, are $(-1, -4, 8)$. Find the coordinates of P_1.

6. Let P_3 be the midpoint of the segment joining the points $P_1(-3, 4, 1)$ and $P_2(-5, 8, 3)$. Find the coordinates of the midpoint of the segment:

    a) joining P_1 and P_3;

    b) joining P_3 and P_2.

1.2 Other Coordinate Systems in $\mathcal{E}_2$ and $\mathcal{E}_3$

Up to now, we used a rectangular coordinate system to specify a point P in the plane $\mathcal{E}_2$ or in the space $\mathcal{E}_3$. There are many situations when the calculations are much simplified by introducing some different coordinate systems.

1.2.1 The Polar Coordinate System (PS)

As an alternative to a rectangular coordinate system (RS) one considers in the plane $\mathcal{E}_2$ a fixed point O, called *pole* and a half-line directed to the right of O, called *polar axis* (see Figure 1.5).

Figure 1.5:

By specifying a directed distance ρ from O to a point P and an angle θ (measured in radians), whose "initial" side is the polar axis and whose "terminal" side is the ray OP, the *polar coordinates* of the point P are (ρ, θ).

One obtains a bijection

$$\mathcal{E}_2 \setminus \{O\} \rightarrow \mathbb{R}_+ \times [0, 2\pi), \quad P \rightarrow (\rho, \theta)$$

which associates to any point P in $\mathcal{E}_2 \setminus \{O\}$ the pair (ρ, θ) (suppose that $O(0,0)$). The positive real number ρ is called the *polar ray* of P and θ is called the *polar angle* of P.

Consider RS to be the rectangular coordinate system in $\mathcal{E}_2$, whose origin O is the pole and whose positive half-axis Ox is the polar axis (see Figure 1.6). The following transformation formulas give the connection between the coordinates of an arbitrary point in the two systems of coordinates.

PS→ RS Let $P(\rho, \theta)$ be a point in the system PS. It is immediate that

$$\begin{cases} x_P = \rho \cos \theta \\ y_P = \rho \sin \theta \end{cases} . \tag{1.6}$$

Figure 1.6:

RS → PS Let $P(x, y)$ be a point in the system RS. It is clear that the polar ray of P is given by the formula

$$\rho = \sqrt{x^2 + y^2}. \tag{1.7}$$

In order to obtain the polar angle of P, it must be considered the quadrant where P is situated. One obtains the following formulas:

Case 1. If $x \neq 0$, then using $\tan\theta = \frac{y}{x}$, one has

$$\theta = \arctan\frac{y}{x} + k\pi, \quad \text{where} \quad k = \begin{cases} 0 \text{ if } P \in I \cup (Ox \\ 1 \text{ if } P \in II \cup III \cup (Ox' \\ 2 \text{ if } P \in IV \end{cases};$$

Case 2. If $x = 0$ and $y \neq 0$, then

$$\theta = \begin{cases} \frac{\pi}{2} \text{ when } P \in (Oy \\ \frac{3\pi}{2} \text{ when } P \in (Oy' \end{cases};$$

Case 3. If $x = 0$ and $y = 0$, then $\theta = 0$.

1.2.2 The Cylindrical Coordinate System (CS)

In order to have a valid coordinate system in the 3-dimensional case, each point of the space must be associated to a unique triple of real numbers (the coordinates of the point) and each triple of real numbers must determine a unique point, as in the case of the rectangular system of coordinates.

Let $P(x, y, z)$ be a point in a rectangular system of coordinates $Oxyz$ and P' be the orthogonal projection of P on the plane xOy. One can associate to the point P the triple (r, θ, z), where (r, θ) are the polar coordinates of P' (see Figure 1.7).

Figure 1.7:

The triple (r, θ, z) gives the *cylindrical coordinates* of the point P. There is the bijection

$$h_1 : \mathcal{E}_3 \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times \mathbb{R},\ P \to (r, \theta, z)$$

and one obtains a new coordinate system, named the *cylindrical coordinate system* (CS) in $\mathcal{E}_3$.

In the following table, the conversion formulas relative to the cylindrical coordinate system (CS) and the rectangular coordinate system (RS) are presented.

Conversion	Formulas
CS→RS $(r, \theta, z) \to (x, y, z)$	$x = r\cos\theta,\ y = r\sin\theta,\ z = z$
RS→CS $(x, y, z) \to (r, \theta, z)$	$r = \sqrt{x^2 + y^2},\ z = z$ and θ is given as follows: Case 1. If $x \neq 0$, then $\theta = \arctan\frac{y}{x} + k\pi,$ where $k = \begin{cases} 0, \text{if } P \in I \cup (Ox \\ 1,\ \text{if } P \in II \cup III \cup (Ox' \\ 2,\ \text{if } P \in IV \end{cases}$ Case 2. If $x = 0$ and $y \neq 0$, then $\theta = \begin{cases} \frac{\pi}{2} \text{ when } P \in (Oy \\ \frac{3\pi}{2} \text{ when } P \in (Oy' \end{cases}$ Case 3. If $x = 0$ and $y = 0$, then $\theta = 0$.

Examples

1) In the cylindrical coordinate system, the equation $r = r_0$ represents a right circular cylinder of radius r_0, centered on the z-axis.

2) The equation $\theta = \theta_0$ describes a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis.

3) The equation $z = z_0$ defines a plane which is parallel to the coordinate plane xOy.

1.2.3 The Spherical Coordinate System (SS)

Another way to associate to each point P in $\mathcal{E}_3$ a triple of real numbers is illustrated in Figure 1.8. If $P(x, y, z)$ is a point in a rectangular system of coordinates $Oxyz$ and P' its orthogonal projection on Oxy, let ρ be the length of the segment $[OP]$, θ be the oriented angle determined by $[Ox$ and $[OP'$ and φ be the oriented angle between $[Oz$ and $[OP$. The triple (ρ, θ, φ) gives the

Figure 1.8:

spherical coordinates of the point P. This way, one obtains the bijection

$$h_2 : \mathcal{E}_3 \setminus \{O\} \to \mathbb{R}_+ \times [0, 2\pi) \times [0, \pi], P \to (\rho, \theta, \varphi),$$

which defines a new coordinate system in $\mathcal{E}_3$, called the *spherical coordinate system* (SS).

The conversion formulas involving the spherical coordinate system (SS) and the rectangular coordinate system (RS) are presented in the following table.

Conversion	Formulas
SS→RS $(\rho, \theta, \varphi) \to (x, y, z)$	$x = \rho \cos\theta \sin\varphi$, $y = \rho \sin\theta \sin\varphi$, $z = \rho \cos\varphi$
RS→SS $(x, y, z) \to (\rho, \theta, \varphi)$	$\rho = \sqrt{x^2 + y^2 + z^2}$, $\varphi = \arccos \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$ θ is given as follows: Case 1. If $x \neq 0$, then $\theta = \arctan \dfrac{y}{x} + k\pi$, where $k = \begin{cases} 0, P' \in I \cup (Ox \\ 1, P' \in II \cup III \cup (Ox' \\ 2, P' \in IV \end{cases}$ Case 2. If $x = 0$ and $y \neq 0$, then $\theta = \begin{cases} \dfrac{\pi}{2}, P' \in (Oy \\ \dfrac{3\pi}{2}, P' \in (Oy' \end{cases}$ Case 3. If $x = 0$ and $y = 0$, then $\theta = 0$

Examples

1) In the spherical coordinate system, the equation $\rho = \rho_0$ represents the set of all points in $\mathcal{E}_3$ whose distance ρ to the origin is ρ_0. This is a sphere of radius ρ_0, centered at the origin.

2) As in the cylindrical coordinates, the equation $\theta = \theta_0$ defines a half-plane attached along the z-axis, making an angle θ_0 with the positive x-axis.

3) The equation $\varphi = \varphi_0$ describes the points P for which the angle determined by $[OP$ and $[Oz$ is φ_0. If $\varphi_0 \neq \dfrac{\pi}{2}$ and $\varphi_0 \neq \pi$, this is a right circular cone, having the vertex at the origin and centered on the z-axis. The equation $\varphi = \dfrac{\pi}{2}$ defines the coordinate plane xOy. The equation $\varphi = \pi$ describes the negative axis $(Oz'$.

1.2.4 Exercises

1. Graph the point P whose polar coordinates are given by:
   $a)(2, \pi)$; $b)(3, \pi/3)$; $c)(4, 3\pi/2)$; $d)(5, \pi/6)$.

2. Find the rectangular coordinates of the points whose polar coordinates are given by:
$a)(1, 2\pi/3)$; $b)(1/2, 7\pi/4)$; $c)(7, \pi/3)$; $d)(\sqrt{3}, 11\pi/6)$.

3. Find the polar coordinates of the points whose rectangular coordinates are given by:
$a)(-3, -3)$; $b)(0, -5)$; $c)(\sqrt{3}, -1)$; $d)(\sqrt{2}, \sqrt{6})$.

4. Find the polar equation corresponding to the given Cartesian equation:

   a) $y = 5$
   b) $x + 1 = 0$
   c) $y = 7x$
   d) $3x + 8y + 6 = 0$
   e) $y^2 = -4x + 4$
   f) $x^2 - 12y - 36 = 0$
   g) $x^2 + y^2 = 36$
   h) $x^2 - y^2 = 25$.

5. Determine, in cylindrical coordinates, the equation of the surface whose equation in rectangular coordinates is
$$z = x^2 + y^2 - 2x + y.$$

6. Find the equation, in rectangular coordinates, of the surface whose equation in cylindrical coordinates is $r = 4\cos\theta$.

7. Find the equation, in spherical coordinates, of the surface whose equation in rectangular coordinates is
$$z = x^2 + y^2.$$

8. Express, in rectangular and spherical coordinates, the following equations, given in cylindrical coordinates:

   a) $r^2 + z^2 = 1$;
   b) $\theta = \dfrac{\pi}{4}$;
   c) $r^2 \cos 2\theta = z$.

9. The equations below are given in spherical coordinates. Express them in rectangular coordinates:

   a) $\rho \sin \varphi = 2 \cos \theta$;
   b) $\rho - 2 \sin \varphi \cos \theta = 0$.

10. Express, in cylindrical and spherical coordinates, the following equations given in rectangular coordinates:

    a) $z = 3x^2 + 3y^2$;
    b) $x^2 + y^2 + z^2 = 2z$.

1.3 Vectors

Let $\mathcal{E}$ denote the Euclidean plane $\mathcal{E}_2$ or the Euclidean 3-space $\mathcal{E}_3$. A pair $(A, B) \in \mathcal{E} \times \mathcal{E}$ is called an *ordered pair* of points or a *vector at the point* A. Such a pair is, shortly, denoted by $\overrightarrow{AB}$. The point A is the *original point*, while B is the *terminal point* and the line AB (if $A \neq B$) gives the direction of $\overrightarrow{AB}$. A vector $\overrightarrow{AB}$ at A has the *orientation* from A to B, i.e. from its original to its terminal point. The length of the segment $[AB]$ represents the *length* of the vector $\overrightarrow{AB}$ and is denoted by $||\overrightarrow{AB}||$ or by $|\overrightarrow{AB}|$. Usually, the vector $\overrightarrow{AB}$ at A is represented as in Figure 1.9. For $A = B$, one obtains the *zero vector*

Figure 1.9:

at the point A.

Define the following relation on $\mathcal{E} \times \mathcal{E}$: $(A, B) \sim (C, D)$ if and only if the segments $[AD]$ and $[BC]$ have the same midpoint. When the points A, B, C and D are not collinear, this means that $(A, B) \sim (C, D)$ if and only if $ABCD$ is a parallelogram.

It is not difficult to check that " $\sim$ " is an equivalence relation. Let us denote by V_3 the set $(\mathcal{E}_3 \times \mathcal{E}_3)/_\sim$ of equivalence classes and by V_2 the set $(\mathcal{E}_2 \times \mathcal{E}_2)/_\sim$.

If $\overrightarrow{AB} \in \mathcal{E} \times \mathcal{E}$, its equivalence class is denoted by $\overline{AB}$ and is called a *vector* in $\mathcal{E}$ ($\mathcal{E}_2$ or $\mathcal{E}_3$). In this case, $\overrightarrow{AB}$ is a *representer* of $\overline{AB}$.

Suppose that $A \neq B$. The line AB defines the *direction* of the vector $\overline{AB}$. The *length* of $\overline{AB}$ is given by

$$||\overline{AB}|| = ||\overrightarrow{AB}|| = AB,$$

the length of the segment $[AB]$. The *orientation* of $\overline{AB}$, from A to B, is given by the orientation of $\overrightarrow{AB}$.

If $A = B$, denote by $\overline{0}$ the *zero vector* represented by $\overrightarrow{AA}$.

Generally, we shall denote the vectors in V_2 or V_3 by small letters: $\overline{a}$, $\overline{b}$,... $\overline{u}$, $\overline{v}$, $\overline{w}$.

Proposition 1.3.1. *Given a vector $\overline{a}$ in V_2 (or V_3) and a fixed point A, there exists a unique representer of $\overline{a}$, having the original point at A.*

Proof: Choose an arbitrary representer $\overrightarrow{CD} \in \overline{a}$ and consider the line $d||CD$, passing through A. Taking into account the orientation, there exists a

Figure 1.10:

unique point B such that $\overrightarrow{AB} \sim \overrightarrow{CD}$ (see Figure 1.10). $\square$

1.3.1 Vector Operations. Components

Let $\overline{a}$ and $\overline{b}$ be two vectors in V_3 (or V_2). The *sum* of $\overline{a}$ and $\overline{b}$ is the vector denoted by $\overline{a} + \overline{b}$, so that, if $\overrightarrow{AB} \in \overline{a}$ and $\overrightarrow{BC} \in \overline{b}$, then $\overrightarrow{AC}$ is the representer of $\overline{a} + \overline{b}$ (see Figure 1.11).

If $\overline{v}$ is a vector in V_3 (or V_2), then the *opposite vector* of $\overline{v}$ is denoted by $-\overline{v}$, so that, if $\overrightarrow{AB}$ is a representer of $\overline{v}$, then $\overrightarrow{BA}$ is a representer of $-\overline{v}$ (see Figure 1.12).

The sum $\overline{a} + (-\overline{b})$ will be, shortly, denoted by $\overline{a} - \overline{b}$ and it will be called the *difference* of the vectors $\overline{a}$ and $\overline{b}$.

Let $\overline{a}$ be a vector in V_3 (or V_2) and k be a real number. The *product* $k \cdot \overline{a}$ is the vector defined as follows:

- $\overline{0}$ if $\overline{a} = \overline{0}$ or $k = 0$;

Figure 1.11:

Figure 1.12:

- if $k > 0$, then $k \cdot \overline{a}$ has the same direction and orientation as $\overline{a}$ and $||k \cdot \overline{a}|| = k \cdot ||\overline{a}||$;
- if $k < 0$, then $k \cdot \overline{a}$ has the same direction as $\overline{a}$, opposite orientation to $\overline{a}$ and $||k \cdot \overline{a}|| = -k \cdot ||\overline{a}||$.

Let $\overline{a}$ be a vector in V_2 and xOy be a rectangular coordinates system in $\mathcal{E}_2$. There exists a unique point $A \in \mathcal{E}_2$, such that $\overrightarrow{OA} \in \overline{a}$ (see Figure 1.13). The coordinates of the point A are called the *components* of the vector $\overline{a}$; we denote it by $\overline{a}(a_1, a_2)$.

Given $\overline{a}$ a vector in V_3 and a rectangular coordinate system $Oxyz$ in $\mathcal{E}_3$, there exists a unique point $A(a_1, a_2, a_3)$, such that $\overrightarrow{OA} \in \overline{a}$. The triple (a_1, a_2, a_3) gives the *components* of $\overline{a}$ and we denote it by $\overline{a}(a_1, a_2, a_3)$.

Since $\overline{0}(0,0)$ in V_2 and $\overline{0}(0,0,0)$ in V_3, then two vectors are equal if and only if they have the same components.

Theorem 1.3.1.1. *Let $\overline{a}(a_1, a_2)$ and $\overline{b}(b_1, b_2)$ be two vectors in V_2 and $k \in \mathbb{R}$. Then:*

Figure 1.13:

1) the components of $\overline{a}+\overline{b}$ *are* (a_1+b_1, a_2+b_2)*;*

2) the components of $k\cdot\overline{a}$ *are* (ka_1, ka_2)*.*

Proof: 1) Let $\overrightarrow{OA}\in\overline{a}$ and $\overrightarrow{OB}\in\overline{b}$ be representers for $\overline{a}$ respectively $\overline{b}$, having the same original point at the origin of the rectangular coordinates system xOy (see Figure 1.14). If $\overrightarrow{OC}\in\overline{a}+\overline{b}$, then $OBCA$ is a parallelogram

Figure 1.14:

and it follows that $c_1 = a_1 + b_1$. In an analogous way, one can show that $c_2 = a_2 + b_2$. $\square$

With the same argument, one obtains:

Theorem 1.3.1.2. *Let $\overline{a}(a_1, a_2, a_3)$ and $\overline{b}(b_1, b_2, b_3)$ be two vectors in V_3 and $k \in \mathbb{R}$. Then:*

1) the components of $\overline{a} + \overline{b}$ are $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$;

2) the components of $k \cdot \overline{a}$ are (ka_1, ka_2, ka_3).

Theorem 1.3.1.3. 1) *If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points in $\mathcal{E}_2$, then*

$$\overline{P_1P_2}(x_2 - x_1, y_2 - y_1).$$

2) *If $Q_1(x_1, y_1, z_1)$ and $Q_2(x_2, y_2, z_2)$ are two points in $\mathcal{E}_3$, then*

$$\overline{Q_1Q_2}(x_2 - x_1, y_2 - y_2, z_2 - z_1).$$

Proof: 2) Let O be the origin of the rectangular coordinates system and remark that

$$\overline{Q_1Q_2} = \overline{Q_1O} + \overline{OQ_2} = \overline{OQ_2} - \overline{OQ_1}.$$

Using Theorem 1.3.1.2, it follows that the vector $\overline{OQ_2} - \overline{OQ_1}$ has the components $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$, therefore

$$\overline{Q_1Q_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1). \quad \square$$

Theorem 1.3.1.4. (properties of summation). *Let $\overline{a}$, $\overline{b}$ and $\overline{c}$ be vectors in V_3 (or V_2) and $\alpha, \beta \in \mathbb{R}$. Then:*

1) $\overline{a} + \overline{b} = \overline{b} + \overline{a}$ (commutativity);

2) $(\overline{a} + \overline{b}) + \overline{c} = \overline{a} + (\overline{b} + \overline{c})$ (associativity);

3) $\overline{a} + \overline{0} = \overline{0} + \overline{a} = \overline{a}$ ($\overline{0}$ is the neutral element for summation);

4) $\overline{a} + (-\overline{a}) = (-\overline{a}) + \overline{a} = \overline{0}$ ($-\overline{a}$ is the symmetrical of $\overline{a}$);

5) $\alpha(\beta\overline{a}) = (\alpha\beta)\overline{a}$;

6) $\alpha \cdot (\overline{a} + \overline{b}) = \alpha \cdot \overline{a} + \beta \cdot \overline{b}$ (distributiveness of multiplication by real scalars with respect to the summation of vectors);

7) $(\alpha + \beta) \cdot \overline{a} = \alpha \cdot \overline{a} + \beta \cdot \overline{a}$ (distributiveness of multiplication by real scalars with respect to the summation of scalars);

8) $1 \cdot \overline{a} = \overline{a}$.

Proof: All the equalities can be proved simply by expressing the components of the vectors on the left and right sides. For instance,

2) If $\overline{a}(a_1, a_2, a_3)$, $\overline{b}(b_1, b_2, b_3)$ and $\overline{c}(c_1, c_2, c_3)$, then the components of the vectors $(\overline{a}+\overline{b})+\overline{c}$ and $\overline{a}+(\overline{b}+\overline{c})$ are $((a_1+b_1)+c_1, (a_2+b_2)+c_2, (a_3+b_3)+c_3)$ respectively $(a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), a_3 + (b_3 + c_3))$ and the conclusion follows. □

Proposition 1.3.1.5. 1) *Let $\overline{a}(a_1, a_2)$ be a vector in V_2. The length of $\overline{a}$ is given by*

$$||\overline{a}|| = \sqrt{a_1^2 + a_2^2}.$$

2) *Let $\overline{a}(a_1, a_2, a_3)$ be a vector in V_3. The length of $\overline{a}$ is given by*

$$||\overline{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^3}.$$

Proof: 2) If $\overrightarrow{OA}$ is the representer of $\overline{a}$ having the original point at the origin of the rectangular coordinate system, then $O(0, 0, 0)$, $A(a_1, a_2, a_3)$ and

$$||\overline{a}|| = ||\overrightarrow{OA}|| = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}. \quad \square$$

- The vectors $\overline{i}(1, 0)$ and $\overline{j}(0, 1)$ in V_2 are called the *unit vectors* (or *versors*) of the coordinate axes Ox and Oy.
- The vectors $\overline{i}(1, 0, 0)$, $\overline{j}(0, 1, 0)$ and $\overline{k}(0, 0, 1)$ are called the *unit vectors* (or *versors*) of the coordinate axes Ox, Oy and Oz (see Figure 1.15).

It is clear that

$$||\overline{i}|| = ||\overline{j}|| = ||\overline{k}|| = 1.$$

We have defined the operations

$$+ : V_2 \times V_2 \to V_2, \quad (\overline{a}, \overline{b}) \mapsto \overline{a} + \overline{b}$$

$$\cdot : \mathbb{R} \times V_2 \to V_2, \quad (k, \overline{a}) \mapsto k \cdot \overline{a}$$

respectively

$$+ : V_3 \times V_3 \to V_3, \quad (\overline{a}, \overline{b}) \mapsto \overline{a} + \overline{b}$$

$$\cdot : \mathbb{R} \times V_3 \to V_3, \quad (k, \overline{a}) \mapsto k \cdot \overline{a}.$$

The following result points out the important algebraic structure of V_2 and V_3.

Figure 1.15:

Theorem 1.3.1.6. 1) $(V_2, +, \cdot)$ *is a vector space over* $\mathbb{R}$, *which is isomorphic to* $(\mathbb{R}^2, +, \cdot)$. *The set* $\{\bar{i}, \bar{j}\}$ *is a base of* V_2, *therefore* $dim_{\mathbb{R}} V_2 = 2$.

2) $(V_3, +, \cdot)$ *is a vector space over* $\mathbb{R}$, *which is isomorphic to* $(\mathbb{R}^3, +, \cdot)$. *The set* $\{\bar{i}, \bar{j}, \bar{k}\}$ *is a base of* V_3, *therefore* $dim_{\mathbb{R}} V_3 = 3$.

Proof: 2) Theorem 1.3.1.4 contains the axioms of a vector space. The map $\psi : V_3 \to \mathbb{R}^3$, given by $\psi(\overline{a}) = (a_1, a_2, a_3)$, where $\overline{a}$ has the components (a_1, a_2, a_3), is bijective and it satisfies

$$\psi(\alpha\overline{a} + \beta\overline{b}) = \alpha\psi(\overline{a}) + \beta\psi(\overline{b}), \quad \forall\ \overline{a}, \overline{b} \in V_3 \quad \text{and} \quad \alpha, \beta \in \mathbb{R}.$$

Indeed, if $\overline{a}(a_1, a_2, a_3)$ and $\overline{b}(b_1, b_2, b_3)$, one has

$$\alpha\overline{a} + \beta\overline{b}(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3),$$

so that

$$\psi(\alpha\overline{a} + \beta\overline{b}) = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3) =$$
$$= \alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3) = \alpha\psi(\overline{a}) + \beta\psi(\overline{b}).$$

Therefore, the map ψ is an isomorphism between V_3 and $\mathbb{R}^3$.

It is well known that $\{e_1, e_2, e_3\}$ is a base in $\mathbb{R}^3$, where $e_1(1, 0, 0)$, $e_2(0, 1, 0)$ and $e_3(0, 0, 1)$. Since $\psi(\bar{i}) = e_1$, $\psi(\bar{j}) = e_2$, $\psi(\overline{k}) = e_3$ and ψ is an isomorphism, then $\{\bar{i}, \bar{j}, \overline{k}\}$ is a base in V_3. □

The base $\{\bar{i}, \bar{j}, \overline{k}\}$ is called the *canonical base* of V_3.

Remark: The vector space V_2 is isomorphic to the 2-dimensional subspace of V_3 given by

$$S = \{\overline{a}(a_1, a_2, a_3) : a_3 = 0\}$$

and an isomorphism is $\phi : V_2 \to S$, $\phi(\overline{a}) = \overline{a}'$, where $\overline{a}(a_1, a_2)$ and $\overline{a}'(a_1, a_2, 0)$.

- Let $\overline{a}$ and $\overline{b}$ be two nonzero vectors in V_3 (or V_2). They are *linearly dependent* if there exist the scalars $\alpha, \beta \in \mathbb{R}^*$ such that $\alpha\overline{a} + \beta\overline{b} = \overline{0}$.

- Let set $\overline{a}$, $\overline{b}$ and $\overline{c}$ be three nonzero vectors in V_3. They are *linearly dependent* if there exist the scalars $\alpha, \beta, \gamma \in \mathbb{R}$, not all equal to zero, such that $\alpha\overline{a} + \beta\overline{b} + \gamma\overline{c} = \overline{0}$.

- The vectors $\overline{a}$ and $\overline{b}$ in V_3 (or V_2), $\overline{a}, \overline{b} \neq \overline{0}$, are *collinear* if they have representers situated on the same line.

- The vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ in V_3, $\overline{a}, \overline{b}, \overline{c} \neq \overline{0}$ are *coplanar* if they have representers situated in the same plane.

Theorem 1.3.1.7. 1) *The vectors $\overline{a}$ and $\overline{b}$ are linearly dependent if and only if they are collinear.*

2) *The vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent in V_3 if and only if they are coplanar.*

Proof: 1) If the vectors $\overline{a}$ and $\overline{b}$ are collinear, then there exists a scalar $\alpha \in \mathbb{R}^*$ such that $\overline{a} = \alpha \cdot \overline{b}$, i.e.

$$1 \cdot \overline{a} + (-\alpha) \cdot \overline{b} = \overline{0},$$

so that $\overline{a}$ and $\overline{b}$ are linearly dependent.

Conversely, if $\alpha\overline{a} + \beta\overline{b} = \overline{0}$ for some scalars α and β, then $\overline{a} = \left(-\dfrac{\beta}{\alpha}\right)\overline{b}$, so that $\overline{a}$ and $\overline{b}$ are collinear.

2) Suppose that the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent. Then, there exist $\alpha, \beta, \gamma \in \mathbb{R}$ not all zero, such that $\alpha\overline{a} + \beta\overline{b} + \gamma\overline{c} = \overline{0}$. Suppose that $\gamma \neq 0$. One obtains

$$\overline{c} = \left(-\frac{\alpha}{\gamma}\right)\overline{a} + \left(-\frac{\beta}{\gamma}\right)\overline{b}.$$

If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are representers of $\overline{a}$ respectively $\overline{b}$, then the representer $\overrightarrow{OC}$ of $\overline{c}$, constructed as in Figure 1.16, is coplanar with $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Conversely, if $\overline{a}$, $\overline{b}$ and $\overline{c}$ are coplanar, let us consider the representers $\overrightarrow{OA} \in \overline{a}$, $\overrightarrow{OB} \in \overline{b}$ and $\overrightarrow{OC} \in \overline{c}$, situated in the same plane. Let $OMCN$ be the parallelogram constructed as in Figure 1.17. Then, there exist $\alpha, \beta \in \mathbb{R}$ such that $\overrightarrow{OM} = \alpha \cdot \overrightarrow{OA}$ and $\overrightarrow{ON} = \beta \cdot \overrightarrow{OB}$. Hence $\overline{OC} = \overline{OM} + \overline{ON} = \alpha \cdot \overline{OA} + \beta \cdot \overline{OB}$ and $\overline{c} = \alpha \cdot \overline{a} + \beta \cdot \overline{b}$, so that $\alpha \cdot \overline{a} + \beta \cdot \overline{b} + (-1) \cdot \overline{c} = \overline{0}$ and the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent. $\square$

Figure 1.16:

Figure 1.17:

Corollary 1.3.1.8. 1) *The set* $\{\overline{a}, \overline{b}\}$ *is a base in* V_2 *if and only if the vectors* $\overline{a}$, $\overline{b}$ *are not collinear.*

2) *The set* $\{\overline{a}, \overline{b}, \overline{c}\}$ *is a base in* V_3 *if and only if the vectors* $\overline{a}$, $\overline{b}$, $\overline{c}$ *are not coplanar.*

1.3.2 Dot product. Projections

One defines the *angle* determined by two nonzero vectors $\overline{a}$ and $\overline{b}$ from V_2 (or V_3), $\theta = \widehat{(\overline{a}, \overline{b})}$, to be the angle determined by their directions, taking into account their orientations, such that $\theta \in [0, \pi]$ (see Figure 1.18)

Figure 1.18:

Now, given the vectors $\overline{a}$ and $\overline{b}$ in V_2 (or V_3), their *dot product* is the real

number defined through

$$\overline{a} \cdot \overline{b} = \begin{cases} |\overline{a}||\overline{b}| \cos\theta, & \text{if } \overline{a} \neq 0,\ \overline{b} \neq 0 \\ 0, & \text{otherwise} \end{cases}. \tag{1.8}$$

Theorem 1.3.2.1. 1) *If $\overline{a}(a_1, a_2)$ and $\overline{b}(b_1, b_2)$ are two vectors in V_2, then*

$$\overline{a} \cdot \overline{b} = a_1 b_1 + a_2 b_2; \tag{1.9}$$

2) *If $\overline{a}(a_1, a_2, a_3)$ and $\overline{b}(b_1, b_2, b_3)$ are two vectors in V_3, then*

$$\overline{a} \cdot \overline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{1.10}$$

Proof: Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be representers of $\overline{a}$, respectively $\overline{b}$, with the same original point O (see Figure 1.19). The cosine theorem in the triangle ΔOAB

Figure 1.19:

gives:

$$AB^2 = |\overline{a}|^2 + |\overline{b}|^2 - 2|\overline{a}||\overline{b}| \cos\theta,$$

hence

$$\overline{a} \cdot \overline{b} = |\overline{a}||\overline{b}| \cos\theta = \frac{1}{2}(|\overline{a}|^2 + |\overline{b}|^2 - AB^2) = \frac{1}{2}(|\overline{a}|^2 + |\overline{b}|^2 - |\overline{AB}|^2) =$$

$$= \frac{1}{2}[(a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - (b_1 - a_1)^2 - (b_2 - a_2)^2 - (b_3 - a_3)^2] =$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3. \quad \square$$

Since $\cos\theta = \dfrac{\overline{a} \cdot \overline{b}}{|\overline{a}||\overline{b}|}$, then, for two nonzero vectors $\overline{a}$ and $\overline{b}$, one has

$$\cos \widehat{(\overline{a}, \overline{b})} = \frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2}}, \text{ for } \overline{a}, \overline{b} \in V_2; \tag{1.11}$$

$$\cos \widehat{(\overline{a}, \overline{b})} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}, \text{ for } \overline{a}, \overline{b} \in V_3. \tag{1.12}$$

Theorem 1.3.2.2. *If $\overline{u}$ and $\overline{v}$ are nonzero vectors in V_2 (or V_3) and θ is the angle between them, then*

a) θ is acute if and only if $\overline{u} \cdot \overline{v} > 0$;

b) θ is obtuse if and only if $\overline{u} \cdot \overline{v} < 0$;

c) $\theta = \dfrac{\pi}{2}$ if and only if $\overline{u} \cdot \overline{v} = 0$.

Proof: The sign of the cosine of the angle determined by two vectors coincides with the sign of their dot product. The assertions are immediate. $\square$

Given an arbitrary vector $\overline{u} \in V_3$ and an associated Cartesian system of coordinates, one defines the *director angles* of $\overline{u}$ to be the three angles determined by $\overline{u}$ with the versors of the system of coordinates, respectively: $\alpha = \widehat{(\overline{u}, \overline{i})}$, $\beta = \widehat{(\overline{u}, \overline{j})}$ and $\gamma = \widehat{(\overline{u}, \overline{k})}$.

Figure 1.20:

The values $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the *director cosines* of the vector $\overline{u}$.

Theorem 1.3.2.3. *The director cosines of a vector $\overline{u}(u_1, u_2, u_3) \in V_3$, $\overline{u} \neq \overline{0}$, are*

$$\cos\alpha = \frac{u_1}{|\overline{u}|}, \; \cos\beta = \frac{u_2}{|\overline{u}|}, \; \cos\gamma = \frac{u_3}{|\overline{u}|}. \tag{1.13}$$

Proof: Since $\overline{i}(1,0,0)$, the formula 1.12 gives $\cos\alpha = \dfrac{\overline{u} \cdot \overline{i}}{|\overline{u}||\overline{i}|} = \dfrac{u_1}{|\overline{u}|}$. $\square$

Remark: For any nonzero vector $\overline{u} \in V_3$, $\dfrac{\overline{u}}{|\overline{u}|}$ is a unit vector, called the *versor of* $\overline{u}$ and, obviously,

$$\frac{\overline{u}}{|\overline{u}|} = \cos\alpha \cdot \overline{i} + \cos\beta \cdot \overline{j} + \cos\gamma \cdot \overline{k}, \text{ with } (\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1.$$

Theorem 1.3.2.4. (algebraic properties of the dot product) *Given* $\overline{a}, \overline{b}, \overline{c} \in V_3$ *(or* V_2*) and* $\lambda \in \mathbb{R}$*, one has:*

1) $\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}$ *(commutativity of the dot product);*

2) $\overline{a} \cdot (\overline{b} + \overline{c}) = \overline{a} \cdot \overline{b} + \overline{a} \cdot \overline{c}$ *(distributivity of the dot product with respect to the summation of vectors);*

3) $\lambda(\overline{a} \cdot \overline{b}) = (\lambda\overline{a}) \cdot \overline{b} = \overline{a} \cdot (\lambda\overline{b})$*;*

4) $\overline{a} \cdot \overline{a} = |\overline{a}|^2$*.*

Sometimes it is useful to decompose a vector into a sum of two terms, one of them having a given direction and the other being orthogonal on this direction.

Figure 1.21:

Let $\overline{u}$ and $\overline{b}$ be two nonzero vectors and project (orthogonally) a representer of the vector $\overline{u}$ on a line passing through the original point of this representer and parallel to the direction of $\overline{b}$. One gets the vector $\overline{w}_1$, having the direction of $\overline{b}$ and, by making the difference $\overline{u} - \overline{w}_1$, another vector $\overline{w}_2$, orthogonal on the direction of $\overline{b}$ (see Figure 1.21); $\overline{u} = \overline{w}_1 + \overline{w}_2$.

The vector $\overline{w}_1$ is called the *orthogonal projection of* $\overline{u}$ *on* $\overline{b}$ and it is denoted by $\mathrm{pr}_{\overline{b}}\overline{u}$. The vector $\overline{w}_2$ is called the *vector component of* $\overline{u}$ *orthogonal to* $\overline{b}$ and $\overline{w}_2 = \overline{u} - \mathrm{pr}_{\overline{b}}\overline{u}$.

Theorem 1.3.2.5. *If $\overline{u}$ and $\overline{b}$ are vectors in V_2 or V_3 and $\overline{b} \neq 0$, then*

- *the orthogonal projection of $\overline{u}$ on $\overline{b}$ is $pr_{\overline{b}}\overline{u} = \dfrac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b}$;*
- *the vector component of $\overline{u}$ orthogonal to $\overline{b}$ is $\overline{u}$-$pr_{\overline{b}}\overline{u} = \overline{u} - \dfrac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b}$.*

Proof: Since $\overline{w}_1$ is parallel to $\overline{b}$, there exists a real number k, such that $\overline{w}_1 = k\overline{b}$. Thus,

$$\overline{u} = \overline{w}_1 + \overline{w}_2 = k\overline{b} + \overline{w}_2.$$

Multiplying by the vector $\overline{b}$, one obtains:

$$\overline{u} \cdot \overline{b} = (\overline{w}_1 + \overline{w}_2) \cdot \overline{b} = (k\overline{b} + \overline{w}_2) \cdot \overline{b} = k|\overline{b}|^2,$$

since $\overline{w}_2$ and $\overline{b}$ are orthogonal. Then the constant k is given by $k = \dfrac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2}$, which completes the proof of the theorem. □

The length of the orthogonal projection of the vector $\overline{u}$ on $\overline{b}$ can be obtained as following:

$$|\text{pr}_{\overline{b}}\overline{u}| = \left| \frac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \cdot \overline{b} \right| = \left| \frac{\overline{u} \cdot \overline{b}}{|\overline{b}|^2} \right| |\overline{b}|,$$

which yields

$$|\text{pr}_{\overline{b}}\overline{u}| = \frac{|\overline{u} \cdot \overline{b}|}{|\overline{b}|}. \tag{1.14}$$

Remark: If θ is the angle between $\overline{u}$ and $\overline{b}$, then

$$|\text{pr}_{\overline{b}}\overline{u}| = |\overline{u}||\cos\theta|.$$

1.3.3 Cross Product. Triple Scalar Product

If $\overline{u} = u_1\overline{i} + u_2\overline{j} + u_3\overline{k}$ and $\overline{v} = v_1\overline{i} + v_2\overline{j} + v_3\overline{k}$ are vectors in V_3, then their *cross product* $\overline{u} \times \overline{v}$ is the vector

$$\overline{u} \times \overline{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \overline{i} + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \overline{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \overline{k}, \tag{1.15}$$

or, shortly,

$$\overline{u} \times \overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{1.16}$$

Theorem 1.3.3.1. *If $\overline{u}$ and $\overline{v}$ are two vectors in V_3, then*

1) $\overline{u} \cdot (\overline{u} \times \overline{v}) = 0$ ($\overline{u} \times \overline{v}$ is orthogonal on $\overline{u}$);

2) $\overline{v} \cdot (\overline{u} \times \overline{v}) = 0$ ($\overline{u} \times \overline{v}$ is orthogonal on $\overline{v}$);

3) $|\overline{u} \times \overline{v}|^2 = |\overline{u}|^2 |\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2$ (Lagrange's identity).

Proof: Let $\overline{u}(u_1, u_2, u_3)$ and $\overline{v}(v_1, v_2, v_3)$. The components of $\overline{u} \times \overline{v}$ are

$$\overline{u} \times \overline{v}(u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Hence,

$$\overline{u} \cdot (\overline{u} \times \overline{v}) = u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) = 0,$$

$$\overline{v} \cdot (\overline{u} \times \overline{v}) = v_1(u_2 v_3 - u_3 v_2) + v_2(u_3 v_1 - u_1 v_3) + v_3(u_1 v_2 - u_2 v_1) = 0,$$

and a simple computation will show that

$$|\overline{u} \times \overline{v}| = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

equals to

$$|\overline{u}|^2 |\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2. \ \square$$

Suppose that $\overline{u}$ and $\overline{v}$ are nonzero vectors in V_3. An immediate consequence of the Lagrange's identity is that $|\overline{u}|^2|\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2 \geq 0$, or $|\overline{u} \cdot \overline{v}| \leq |\overline{u}||\overline{v}|$, which leads, after replacing the components of the vectors, to the Cauchy-Schwartz inequality. The equality $|\overline{u} \cdot \overline{v}| = |\overline{u}||\overline{v}|$ holds if and only if the vector $\overline{u} \times \overline{v}$ is the zero vector, i.e. its components are all zero, which happens if and only if $\dfrac{v_1}{u_1} = \dfrac{v_2}{u_2} = \dfrac{v_3}{u_3} = \lambda$, or $\overline{v} = \lambda \overline{u}$, $\lambda \in \mathbb{R}^*$. In summary, one has:

Theorem 1.3.3.2. *If $\overline{u}$ and $\overline{v}$ are nonzero vectors in V_3, then $\overline{u} \times \overline{v} = \overline{0}$ if and only if $\overline{u}$ and $\overline{v}$ are parallel.*

Using the Lagrange's identity and the definition of the dot product, one can determine the length of the cross product of two nonzero vectors from V_3. If θ is the angle determined by $\overline{u}$ and $\overline{v}$, $\theta \in [0, \pi]$, then

$$|\overline{u} \times \overline{v}|^2 = |\overline{u}|^2 |\overline{v}|^2 - (\overline{u} \cdot \overline{v})^2 = |\overline{u}|^2 |\overline{v}|^2 - |\overline{u}|^2 |\overline{v}|^2 \cos^2 \theta = |\overline{u}|^2 |\overline{v}|^2 \sin^2 \theta,$$

hence

$$|\overline{u} \times \overline{v}| = |\overline{u}||\overline{v}| \sin \theta. \tag{1.17}$$

The formula 1.17 has a useful geometric meaning: the length of the cross product of two nonzero vectors from V_3 is exactly the area of the parallelogram constructed on the two vectors (see Figure 1.22).

The vector $\overline{u} \times \overline{v}$ is completely determined by the following three properties:

Figure 1.22:

- $\overline{u} \times \overline{v}$ is orthogonal on both $\overline{u}$ and $\overline{v}$ (hence on the plane determined by two coplanar representers of $\overline{u}$ and $\overline{v}$);
- the orientation of $\overline{u} \times \overline{v}$ is given by the right-hand rule;
- $|\overline{u} \times \overline{v}| = |\overline{u}||\overline{v}| \sin \theta$.

Figure 1.23:

Theorem 1.3.3.3. (algebraic properties of the cross product) *For any vectors $\overline{u}$, $\overline{v}$ and $\overline{w}$ from V_3 and any scalar $\lambda \in \mathbb{R}$, the following equalities hold:*

a) $\overline{u} \times \overline{v} = -\overline{v} \times \overline{u}$;

b) $\overline{u} \times (\overline{v} + \overline{w}) = \overline{u} \times \overline{v} + \overline{u} \times \overline{w}$;

c) $(\overline{u} + \overline{v}) \times \overline{w} = \overline{u} \times \overline{w} + \overline{v} \times \overline{w}$;

d) $\lambda(\overline{u} \times \overline{v}) = (\lambda\overline{u}) \times \overline{v} = \overline{u} \times (\lambda\overline{v})$;

e) $\overline{u} \times \overline{0} = \overline{0} \times \overline{u} = \overline{0}$;

f) $\overline{u} \times \overline{u} = \overline{0}$.

It is very easy to compute the cross products of the versors of the axes:

$$\begin{array}{ccc} \overline{i} \times \overline{j} = \overline{k} & \overline{j} \times \overline{k} = \overline{i} & \overline{k} \times \overline{i} = \overline{j} \\ \overline{j} \times \overline{i} = -\overline{k} & \overline{k} \times \overline{j} = -\overline{i} & \overline{i} \times \overline{k} = -\overline{j} \\ \overline{i} \times \overline{i} = \overline{0} & \overline{j} \times \overline{j} = \overline{0} & \overline{k} \times \overline{k} = \overline{0} \end{array} .$$

Given three vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ from V_3, one defines their *triple scalar product* to be the real number $(\overline{a}, \overline{b}, \overline{c}) = \overline{a} \cdot (\overline{b} \times \overline{c})$.

If $\overline{a} = (a_1, a_2, a_3)$, $\overline{b} = (b_1, b_2, b_3)$ and $\overline{c} = (c_1, c_2, c_3)$, then the triple scalar product can be calculated as

$$(\overline{a}, \overline{b}, \overline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . \tag{1.18}$$

Indeed,

$$(\overline{a}, \overline{b}, \overline{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) =$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} .$$

Remark: It can be seen easily that the triple scalar product can be also seen as $(\overline{a}, \overline{b}, \overline{c}) = (\overline{a} \times \overline{b}) \cdot \overline{c}$.

Theorem 1.3.3.4. (properties of the triple scalar product) *If $\overline{a}$, $\overline{b}$ and $\overline{c}$ are vectors in V_3, then:*

a) $(\overline{a}, \overline{b}, \overline{c}) = (\overline{c}, \overline{a}, \overline{b}) = (\overline{b}, \overline{c}, \overline{a})$;

b) $(\overline{a}, \overline{b}, \overline{c}) = 0$ *if and only if $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent (i.e. they have representers situated on the same plane).*

Proof: a) It follows immediately from the properties of the determinants;

b) The triple scalar product $(\overline{a}, \overline{b}, \overline{c}) = 0$ if and only if the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$, therefore its rows, for instance, are linearly dependent, meaning that the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly dependent, so that one can choose a representer for each, having the same original point, which are contained into a plane. □

The triple scalar product has a geometric meaning. Suppose that the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ are linearly independent and choose a representer for each, having the same original point. These form the adjacent sides of a parallelepiped (see Figure 1.24). Suppose that the base of this parallelepiped is

Figure 1.24:

the parallelogram constructed on $\overline{b}$ and $\overline{c}$. The height of the parallelepiped is the length of the orthogonal projection of the vector $\overline{a}$ on the direction of the vector $\overline{b} \times \overline{c}$,

$$h = |\mathrm{pr}_{\overline{b}\times\overline{c}}\overline{a}| = \left|\frac{\overline{a}\cdot(\overline{b}\times\overline{c})}{|\overline{b}\times\overline{c}|}\right| = \frac{|(\overline{a},\overline{b},\overline{c})|}{|\overline{b}\times\overline{c}|}.$$

Then, the volume of the parallelepiped whose adjacent sides are the vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$ is the absolute value of the triple scalar product $(\overline{a},\overline{b},\overline{c})$:

$$V = h\cdot \mathrm{Area}(\overline{b},\overline{c}) = \frac{|(\overline{a},\overline{b},\overline{c})|}{|\overline{b}\times\overline{c}|}|\overline{b}\times\overline{c}| = |(\overline{a},\overline{b},\overline{c})|. \tag{1.19}$$

1.3.4 Exercises

1) Let M and N be the midpoints of two opposite sides of a quadrilateral $ABCD$ and let P be the midpoint of $[MN]$. Prove that

$$\overline{PA}+\overline{PB}+\overline{PC}+\overline{PD}=\overline{0}.$$

2) In a circle of center O, let M be the intersection point of two perpendicular chords $[AB]$ and $[CD]$. Show that

$$\overline{OA}+\overline{OB}+\overline{OC}+\overline{OD}=2\overline{OM}.$$

3) Consider, in the 3-dimensional space, the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Prove that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a new parallelogram.

4) Let ABC be a triangle, H its orthocenter, O the circumcenter (the center of its circumscribed circle), G the center of gravity of the triangle and A' diametrically opposed to A (in the circumscribed circle). Then :

   a) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OH}$ (Sylvester's formula)
   b) $\overline{HB} + \overline{HC} = \overline{HA'}$
   c) $\overline{HA} + \overline{HB} + \overline{HC} = 2\overline{HO}$
   d) $\overline{HA} + \overline{HB} + \overline{HC} = 3\overline{HG}$
   e) the points H, G, O are collinear and $2GO = HG$ (the Euler's straight line of the triangle).

5) Let ABC be a triangle and a, b, c the lengths of its sides, respectively. If A_1 is the intersection point of the internal bisector of the angle $\hat{A}$ and BC and M is an arbitrary point, then

$$\overline{MA_1} = \frac{b}{b+c}\overline{MB} + \frac{c}{b+c}\overline{MC}.$$

6) The midpoints of the diagonals of a complete quadrilateral are collinear (the Newton-Gauss' straight line).

7) *(Cauchy-Buniakovski-Schwarz)* If $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, then

$$(a_1b_1 + a_2b_2 + a_2c_2)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

8) *(Cosine theorem in the space)* For a tetrahedron $ABCD$,

$$\cos(\widehat{\overline{AB}, \overline{CD}}) = \frac{AD^2 + BC^2 - AC^2 - BD^2}{2AB \cdot CD}.$$

9) *(Median line theorem in the space)* Let $ABCD$ be a tetrahedron and G_A the center of gravity of the BCD side. Then the following equality holds:

$$9AG_A^2 = 3(AB^2 + AC^2 + AD^2) - (BC^2 + CD^2 + BD^2).$$

10) Let ΔABC and $\Delta A'B'C'$ be two triangles in the same plane, so that the perpendicular lines through A, B, C on $B'C'$, $C'A'$ respectively $A'B'$ are concurrent. Then so are the perpendicular lines through A', B', C' on BC, CA respectively AB.

11) Find a vector orthogonal on both $\overline{u}$ and $\overline{v}$.

    a) $\overline{u} = -7\overline{i} + 3\overline{j} + \overline{k}$, $\overline{v} = 2\overline{i} + 4\overline{k}$

    b) $\overline{u} = (-1, -1, -1)$, $\overline{v} = (2, 0, 2)$.

12) Let $\overline{a}$, $\overline{b}$ and $\overline{c}$ be three noncollinear vectors. Show that there exists a triangle ABC with $\overline{BC} = \overline{a}$, $\overline{CA} = \overline{b}$ and $\overline{AB} = \overline{c}$ if and only if

$$\overline{a} \times \overline{b} = \overline{b} \times \overline{c} = \overline{c} \times \overline{a}.$$

13) Find the area of the triangle having the vertices $A(1, 0, 1)$, $B(0, 2, 3)$, and $C(2, 1, 0)$.

14) Prove that:

    a) $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{a} \cdot \overline{b}) \cdot \overline{c} = \begin{vmatrix} \overline{b} & \overline{c} \\ \overline{a} \cdot \overline{b} & \overline{a} \cdot \overline{c} \end{vmatrix}$

    b) $(\overline{a} \times \overline{b}) \times \overline{c} = (\overline{a} \cdot \overline{c}) \cdot \overline{b} - (\overline{b} \cdot \overline{c}) \cdot \overline{a} = \begin{vmatrix} \overline{b} & \overline{a} \\ \overline{b} \cdot \overline{c} & \overline{a} \cdot \overline{c} \end{vmatrix}$

15) Verify the Laplace's formula:

$$(\overline{a} \times \overline{b}) \cdot (\overline{c} \times \overline{d}) = \begin{vmatrix} \overline{a} \cdot \overline{c} & \overline{a} \cdot \overline{d} \\ \overline{b} \cdot \overline{c} & \overline{b} \cdot \overline{d} \end{vmatrix}.$$

16) Prove that

$$(\overline{a} \times \overline{b}) \times (\overline{c} \times \overline{d}) = (\overline{a}, \overline{c}, \overline{d}) \cdot \overline{b} - (\overline{b}, \overline{c}, \overline{d}) \cdot \overline{a} = (\overline{a}, \overline{b}, \overline{d}) \cdot \overline{c} - (\overline{a}, \overline{b}, \overline{c}) \cdot \overline{d}.$$

17) Show that

$$(\overline{u} \times \overline{v}, \overline{v} \times \overline{w}, \overline{w} \times \overline{u}) = (\overline{u}, \overline{v}, \overline{w})^2.$$

18) The *mutual vectors* of the vectors $\overline{u}$, $\overline{v}$, $\overline{w}$, supposed to be not in the same plane, are

$$\overline{u}' = \frac{\overline{v} \times \overline{w}}{(\overline{u}, \overline{v}, \overline{w})}, \quad \overline{v}' = \frac{\overline{w} \times \overline{u}}{(\overline{u}, \overline{v}, \overline{w})}, \quad \overline{w}' = \frac{\overline{u} \times \overline{v}}{(\overline{u}, \overline{v}, \overline{w})}.$$

a) Find the mutual vectors of $\overline{i}$, $\overline{j}$, $\overline{k}$.

b) If $\overline{a} = x\overline{u} + y\overline{v} + z\overline{w}$, prove that

$$x = \overline{a} \cdot \overline{u}', \quad y = \overline{a} \cdot \overline{v}', \quad z = \overline{a} \cdot \overline{w}'.$$

c) Show that the mutual vectors of $\overline{u}'$, $\overline{v}'$, $\overline{w}'$ are respectively $\overline{u}$, $\overline{v}$, $\overline{w}$.

19) Let a, b and c be the lengths of the sides of the triangle ΔABC and R the radius of its circumscribed circle. Then:

a) $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$;

b) $OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$.

20) If E and F are the midpoints of the diagonals AC and BD of the convexe quadrilateral $ABCD$, then

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4EF^2.$$

21) Let K, L, M and N be the midpoints of the sides AB, BC, CD and respectively DA of the convexe quadrilateral $ABCD$. Then

$$AB^2 + CD^2 - AD^2 - BC^2 = 2(LN^2 - KM^2) = 2AC \cdot BD \cos\varphi,$$

where $\varphi = (\widehat{AC, BD})$.

21) Let ABC be an arbitrary triangle, G its center of gravity, I its incenter, H its orthocenter and O the center of its circumscribed circle. If P is an arbitrary point and $\overline{r}_A = \overline{PA}$, $\overline{r}_B = \overline{PB}$, $\overline{r}_C = \overline{PC}$, then:

a) $\overline{PG} = \dfrac{\overline{r}_A + \overline{r}_B + \overline{r}_C}{3}$;

b) $\overline{PI} = \dfrac{a \cdot \overline{r}_A + b \cdot \overline{r}_B + c \cdot \overline{r}_C}{a + b + c}$;

c) $\overline{PH} = \dfrac{\tan A \cdot \overline{r}_A + \tan B \cdot \overline{r}_B + \tan C \cdot \overline{r}_C}{\tan A + \tan B + \tan C}$;

d) $\overline{PO} = \dfrac{\sin 2A \cdot \overline{r}_A + \sin 2B \cdot \overline{r}_B + \sin 2C \cdot \overline{r}_C}{\sin 2A + \sin 2B + \sin 2C}$.

22) Let d be a line in the plane of the triangle ΔABC and A', B' and C' the orthogonal projections of the vertices A, B, respectively C on d. Prove that the orthogonal lines through A', B' respectively C' on BC, CA respectively AB are concurrent.

23) The radius of the sphere tangent at the point O to the face (BOC) of the tetrahedron $OABC$ and passing through the point A is

$$R = \frac{a \sin \alpha}{2(1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma)^{\frac{1}{2}}},$$

where $a = OA$, $\alpha = \widehat{BOC}$, $\beta = \widehat{COA}$, $\gamma = \widehat{AOB}$.

Chapter 2

Two-Dimensional Analytic Geometry

2.1 Several Equations of Lines

Analytic Geometry allows us to express in an algebraic language geometric properties of objects. We start here with the Euclidean plane, whose basic elements are the points and the lines. We associate a Cartesian system of coordinates, in which an arbitrary point is characterized by two real numbers, its Cartesian coordinates, as we saw in the first chapter. We shall present different equations of lines in a 2-dimensional Euclidean space.

2.1.1 Parametric Equations of Lines

A line d can be determined by specifying a point $P_0(x_0, y_0)$ on the line and a nonzero vector $\overline{v}(a, b)$, parallel to the line (the direction of the line).

Figure 2.1:

The line d in a 2-space, passing through the point $P_0(x_0, y_0)$ and parallel to the nonzero vector $\overline{v}(a, b)$ has the *parametric equations*

$$d : \begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases} \qquad t \in \mathbb{R}. \tag{2.1}$$

Indeed, for any point $P(x, y)$ on the line d, the vectors $\overline{P_0P}$ and $\overline{v}$ are linearly dependent, since they are parallel, hence there exists $t \in \mathbb{R}$ such that $\overline{P_0P} = t\overline{v}$. Identifying the components of these two vectors, respectively, one obtains

$$d : \begin{cases} x - x_0 = at \\ y - y_0 = bt \end{cases}$$

which leads to the formula (2.1).

The vector $\overline{v}$ is said to be the *director vector* of the line d.

2.1.2 Vector Equations of Lines

The vector language can be used to express the parametric equations of the line (2.1) in a shorter form. The line d is again the line passing through P_0 and parallel to the vector $\overline{v}$. Choosing an arbitrary point O in the plane, one can characterize any point P by its *position vector*, i.e. the vector having the original point O and the terminal point P.

Figure 2.2:

As before, the point P belongs to the line d if and only if the vectors $\overline{P_0P}$ and $\overline{v}$ are linearly dependent. This means that there exists $t \in \mathbb{R}$, such that $\overline{P_0P} = t\overline{v}$. But $\overline{P_0P} = \overline{OP} - \overline{OP_0} = \overline{r} - \overline{r}_0$ (see Figure 3.4), hence $t\overline{v} = \overline{r} - \overline{r}_0$, and the *vector equation* of the line passing through P_0 and of director vector $\overline{v}$ is

$$\overline{r} = \overline{r_0} + t\overline{v}. \tag{2.2}$$

2.1.3 Symmetric Equations of Lines

If, in (2.1), one expresses twice the parameter t, one obtains the *symmetric equation* of the line d passing through the point $P_0(x_0, y_0)$ and of director vector $\overline{v}(a, b)$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}. \tag{2.3}$$

Remark: The vector $\overline{v}$ is a nonzero vector, so that at least one of the denominators a and b is different from zero. Suppose that $a = 0$. Then, (2.1) becomes $\begin{cases} x = x_0 \\ y = y_0 + bt \end{cases}$. In fact, $\dfrac{x - x_0}{0} = \dfrac{y - y_0}{b}$ is just a convenient way to write that the numerator is zero when the denominator is zero. This is the particular case of a line which is parallel to Oy. If $b = 0$, one obtains the equations of a line parallel to Ox.

2.1.4 General Equations of Lines

A simple computation shows that (2.3) can be written in the form

$$Ax + By + C = 0, \qquad \text{with} \quad A^2 + B^2 \neq 0, \tag{2.4}$$

meaning that any line from the 2-space is characterized by a first degree equation. Conversely, such of an equation represents a line, since the formula (2.4) is equivalent to $\dfrac{x + \frac{C}{A}}{-\frac{B}{A}} = \dfrac{y}{1}$ and this is the symmetric equation of the line passing through $P_0\left(-\dfrac{C}{A}, 0\right)$ and parallel to $\overline{v}\left(-\dfrac{B}{A}, 1\right)$.

The equation (2.4) is called *general equation* of the line.

2.1.5 Reduced Equations of Lines

Consider a line given by its general equation $Ax + By + C = 0$, where at least one of the coefficients A and B is nonzero. One may suppose that $B \neq 0$, so that the equation can be divided by B. One obtains

$$y = mx + n \tag{2.5}$$

which is said to be the *reduced equation* of the line.

Remark: If $B = 0$, (2.4) becomes $Ax + C = 0$, or $x = -\dfrac{C}{A}$, a line parallel to Oy. (In the same way, if $A = 0$, one obtains the equation of a line parallel to Ox).

Let d be a line of equation $y = mx+n$ in a Cartesian system of coordinates and suppose that the line is not parallel to Oy. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two different points on d and φ be the angle determined by d and Ox (see Figure 2.3); $\varphi \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.

Figure 2.3:

The points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ belong to d, hence $\begin{cases} y_1 = mx_1 + n \\ y_2 = mx_2 + n \end{cases}$, and $x_2 \neq x_1$, since d is not parallel to Oy. Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \varphi. \tag{2.6}$$

The number $m = \tan \varphi$ is called the *angular coefficient* of the line d.

It is immediate that the equation of the line passing through the point $P_0(x_0, y_0)$ and of the given angular coefficient m is

$$y - y_0 = m(x - x_0). \tag{2.7}$$

2.1.6 Equations of Lines Determined by Two Points

A line can be uniquely determined by two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line. The line can be seen to be the line passing through the point $P_1(x_1, y_1)$ and having $\overline{P_1P_2}(x_2 - x_1, y_2 - y_1)$ as director vector, therefore its equation is

$$d : \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \tag{2.8}$$

The equation (2.8) can be put in the form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{2.9}$$

Given three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$, they are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

2.1.7 Exercises

1) The sides $[BC]$, $[CA]$, $[AB]$ of the triangle ΔABC are divided by the points M, N respectively P into the same ratio k. Prove that the triangles ΔABC and ΔMNP have the same center of gravity.

2) Sketch the graph of $x^2 - 4xy + 3y^2 = 0$.

3) Find the equation of the line passing through the intersection point of the lines

$$d_1 : 2x - 5y - 1 = 0, \quad d_2 : x + 4y - 7 = 0$$

and through a point M which divides the segment $[AB]$, $A(4, -3)$, $B(-1, 2)$, into the ratio $k = \dfrac{2}{3}$.

4) Let A be a mobile point on the Ox axis and B a mobile point on Oy, so that $\dfrac{1}{OA} + \dfrac{1}{OB} = k$ (constant). Prove that the lines AB pass through a fixed point.

5) Find the equation of the line passing through the intersection point of

$$d_1 : 3x - 2y + 5 = 0, \quad d_2 : 4x + 3y - 1 = 0$$

and crossing the Oy axis at the point A with $OA = 3$.

6) Find the parametric equations of the line through P_1 and P_2, when

   a) $P_1(3, -2)$, $P_2(5, 1)$;
   b) $P_1(4, 1)$, $P_2(4, 3)$.

7) Find the parametric equations of the line through $P(-5, 2)$ and parallel to $\overline{v}(2, 3)$.

8) Show that the equations

$$x = 3 - t, y = 1 + 2t \qquad \text{and} \qquad x = -1 + 3t, y = 9 - 6t$$

represent the same line.

9) Find the vector equation of the line passing through P_1 and P_2, when

   a) $P_1(2, -1)$, $P_2(-5, 3)$;

   b) $P_1(0, 3)$, $P_2(4, 3)$.

2.2 Parallelism and Orthogonality

2.2.1 Intersection of Two Lines

Let $d_1 : a_1x + b_1y + c_1 = 0$ and $d_2 : a_2x + b_2y + c_2 = 0$ be two lines in $\mathcal{E}_2$. The solution of the system of equation

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

will give the set of the intersection points of d_1 and d_2.

1) If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, the system has a unique solution (x_0, y_0) and the lines have a unique intersection point $P_0(x_0, y_0)$. They are *secant.*

2) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, the system is not compatible, and the lines have no points in common. They are *parallel.*

3) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the system has an infinity of solutions, and the lines coincide. They are *identical.*

If $d_i : a_ix + b_iy + c_i = 0$, $i = \overline{1, 3}$ are three lines in $\mathcal{E}_2$, then they are concurrent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{2.10}$$

2.2.2 Bundle of Lines

The set of all the lines passing through a given point P_0 is said to be a *bundle* of lines. The point P_0 is called the *vertex* of the bundle.

If the point P_0 is of coordinates $P_0(x_0, y_0)$, then the equation of the bundle of vertex P_0 is

$$r(x - x_0) + s(y - y_0) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{2.11}$$

Figure 2.4:

Remark: One may suppose that $s \neq 0$ and divide in (2.11) by s. One obtains the *reduced equation* of the bundle,

$$y - y_0 = m(x - x_0), \qquad m \in \mathbb{R}, \tag{2.12}$$

in which the line $x = x_0$ is missing. Analogously, if $r \neq 0$, one obtains the bundle, except the line $y = y_0$.

If the point P_0 is given as the intersection of two lines, then its coordinates are the solution of the system

$$\begin{cases} d_1 : a_1x + b_1y + c_1 = 0 \\ d_2 : a_2x + b_2y + c_2 = 0 \end{cases},$$

supposed to be compatible. The equation of the bundle of lines through P_0 is

$$r(a_1x + b_1y + c_1) + s(a_2x + b_2y + c_2) = 0, \qquad (r, s) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \tag{2.13}$$

Remark: As before, if $r \neq 0$ (or $s \neq 0$), one obtains the reduced equation of the bundle, containing all the lines through P_0, except d_1 (respectively d_2).

2.2.3 The Angle of Two Lines

Let d_1 and d_2 be two concurrent lines, given by their reduced equations:

$$d_1 : y = m_1 x + n_1 \quad \text{and} \quad d_2 : y = m_2 x + n_2.$$

The angular coefficients of d_1 and d_2 are $m_1 = \tan\varphi_1$ and $m_2 = \tan\varphi_2$ (see Figure 2.5). One may suppose that $\varphi_1 \neq \frac{\pi}{2}$, $\varphi_2 \neq \frac{\pi}{2}$, $\varphi_2 \geq \varphi_1$, such that $\varphi = \varphi_2 - \varphi_1 \in [0, \pi] \setminus \{\frac{\pi}{2}\}$.

Figure 2.5:

The angle determined by d_1 and d_2 is given by

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2},$$

hence

$$\tan\varphi = \frac{m_2 - m_2}{1 + m_1 m_2}. \tag{2.14}$$

1) The lines d_1 and d_2 are parallel if and only if $\tan\varphi = 0$, therefore

$$d_1 \parallel d_2 \iff m_1 = m_2. \tag{2.15}$$

2) The lines d_1 and d_2 are orthogonal if and only if they determine an angle of $\frac{\pi}{2}$, hence

$$d_1 \perp d_2 \iff m_1 m_2 + 1 = 0. \tag{2.16}$$

2.2.4 Exercises

1) Given the line $d: 2x+3y+4=0$, find the equation of a line d_1 passing through the point $M_0(2,1)$, in the following situations:

   a) d_1 is parallel with d;

   b) d_1 is orthogonal on d;

   c) the angle determined by d and d_1 is $\varphi = \dfrac{\pi}{4}$.

2) The vertices of the triangle ΔABC are the intersection points of the lines

$$d_1 : 4x+3y-5=0, \quad d_2 : x-3y+10=0, \quad d_3 : x-2=0.$$

   1. Find the coordinates of A, B, C.
   2. Find the equations of the median lines of the triangle.
   3. Find the equations of the heights of the triangle.

3) Find the coordinates of the symmetrical of the point $P(-5,13)$ with respect to the line $d: 2x-3y-3=0$.

4) Find the coordinates of the point P on the line $d: 2x-y-5=0$ for which the sum $AP+PB$ is minimum, when $A(-7,1)$ and $B(-5,5)$.

5) Find the coordinates of the circumcenter (the center of the circumscribed circle) of the triangle determined by the lines $4x-y+2=0$, $x-4y-8=0$ and $x+4y-8=0$.

6) Prove that, in any triangle ΔABC, the orthocenter H, the center of gravity G and the circumcenter O are collinear.

7) Given the bundle of lines of equations $(1-t)x+(2-t)y+t-3=0$, $t \in \mathbb{R}$ and $x+y-1=0$, find:

   a) the coordinates of the vertex of the bundle;

   b) the equation of the line in the bundle which cuts Ox and Oy in M respectively N, such that $OM^2 \cdot ON^2 = 4(OM^2+ON^2)$.

8) Let $\mathcal{B}$ be the bundle of vertex $M_0(5,0)$. An arbitrary line from $\mathcal{B}$ intersects the lines $d_1 : y-2=0$ and $d_2 : y-3=0$ in M_1 respectively M_2. Prove that the line passing through M_1 and parallel to OM_2 passes through a fixed point.

9) The vertices of the quadrilateral $ABCD$ are $A(4,3)$, $B(5,-4)$, $C(-1,-3)$ and $D((-3,-1)$.

   a) Find the coordinates of the points $\{E\} = AB \cap CD$ and $\{F\} = BC \cap AD$;

   b) Prove that the midpoints of the segments $[AC]$, $[BD]$ and $[EF]$ are collinear.

10) Let M be a point whose coordinates satisfy

$$\frac{4x+2y+8}{3x-y+1} = \frac{5}{2}.$$

   a) Prove that M belongs to a fixed line;

   b) Find the minimum of $x^2 + y^2$, when $M \in d \setminus \{M_0(-1,-2)\}$.

11) Find the geometric locus of the points whose distances to two orthogonal lines have a constant ratio.

Chapter 3

Three-Dimensional Analytic Geometry

In a 3-dimensional Euclidean space $\mathcal{E}_3$, endowed with a rectangular system of coordinates $Oxyz$, a point $P \in \mathcal{E}_3$ is characterized by three real numbers, the coordinates of the point, $P(x, y, z)$. We study in this chapter the planes and the lines in the 3-dimensional space.

3.1 Analytic Representation of Planes

A plane π in the 3-dimensional space is uniquely determined by specifying a point $P_0(x_0, y_0, z_0)$ in the plane and a nonzero vector $\overline{n}(a, b, c)$, orthogonal to the plane. $\overline{n}$ is called the *normal vector* to the plane π.

Figure 3.1:

An arbitrary point $P(x, y, z)$ is contained into the plane π if and only if

$\overline{n} \perp \overline{P_0P}$, or $\overline{n} \cdot \overline{P_0P} = 0$. But $\overline{P_0P}(x - x_0, y - y_0, z - z_0)$ and one obtains the *normal* equation of the plane π containing the point $P_0(x_0, y_0, z_0)$ and of normal vector $\overline{n}(a, b, c)$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{3.1}$$

Remark: The equation (3.1) can be written in the form $ax+by+cz+d = 0$.

Theorem 3.1.1. *Given $a, b, c, d \in \mathbb{R}$, with $a^2 + b^2 + c^2 > 0$, the equation*

$$ax + by + cz + d = 0 \tag{3.2}$$

represents a plane, having $\overline{n}(a, b, c)$ as normal vector.

Proof: One may suppose that $a \neq 0$. The equation (3.2) can be put in the form $a\left(x + \dfrac{d}{a}\right) + by + cz = 0$, which represents the plane containing the point $P\left(-\dfrac{d}{a}, 0, 0\right)$ and having $\overline{n}(a, b, c)$ as normal vector. $\square$

The equation (3.2) is called the *general* equation of the plane.

Given a fixed point O in the 3-space, any point P is characterized by its position vector $\overline{r}_P = \overline{OP}$.

Theorem 3.1.2. *a) The vector equation of the plane π, determined by three noncollinear points A, B and C, is*

$$\overline{r} = (1 - \alpha - \beta)\overline{r}_A + \alpha\overline{r}_B + \beta\overline{r}_C, \qquad \alpha, \beta \in \mathbb{R}. \tag{3.3}$$

b) The vector equation of the plane π, determined by a point A and two non-parallel directions $\overline{v}_1$ and $\overline{v}_2$ contained into the plane, is

$$\overline{r} = \overline{r}_A + \alpha\overline{v}_1 + \beta\overline{v}_2, \qquad \alpha, \beta \in \mathbb{R}. \tag{3.4}$$

Proof: a) For any arbitrary point $M \in \pi$, the vectors $\overline{AB}$, $\overline{AC}$ and $\overline{AM}$ are linearly dependent, since they are coplanar (see Figure 3.2 a), i.e. there exist the scalars α and $\beta \in \mathbb{R}$, such that

$$\overline{AM} = \alpha\overline{AB} + \beta\overline{AC}.$$

By expressing each vector in this equality, one gets

$$\overline{r}_M - \overline{r}_A = \alpha(\overline{r}_B - \overline{r}_A) + \beta(\overline{r}_C - \overline{r}_A),$$

Figure 3.2:

or, equivalently,

$$\overline{r}_M = (1-\alpha-\beta)\overline{r}_A + \alpha\overline{r}_B + \beta\overline{r}_C.$$

b) Similarly, for any point $M \in \pi$, the vectors $\overline{AM}$, $\overline{v}_1$ and $\overline{v}_2$ are linearly dependent (see Figure 3.2 b), such that there exist α, $\beta \in \mathbb{R}$, with

$$\overline{AM} = \alpha\overline{v}_1 + \beta\overline{v}_2,$$

or

$$\overline{r}_M - \overline{r}_A = \alpha\overline{v}_1 + \beta\overline{v}_2$$

and

$$\overline{r}_M = \overline{r}_A + \alpha\overline{v}_1 + \beta\overline{v}_2. \qquad \square$$

If the points A, B and C which determine the plane π are of coordinates $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$ and an arbitrary point of π is $M(x, y, z)$, then the equation (3.3) decomposes into three linear equations:

$$\begin{cases} x = (1-\alpha-\beta)x_A + \alpha x_B + \beta x_C \\ y = (1-\alpha-\beta)y_A + \alpha y_B + \beta y_C \\ z = (1-\alpha-\beta)z_A + \alpha z_B + \beta z_C \end{cases}.$$

This system must have an infinity of solutions (α, β), so that

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0, \tag{3.5}$$

which is the analytic equation of the plane determined by three noncollinear points.

The points A, B, C and D are coplanar if and only if:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0. \tag{3.6}$$

Replacing now, in (3.4), the vectors $\overline{v}_1(p_1, q_1, r_1)$ and $\overline{v}_2(p_2, q_2, r_2)$ and the points $A(x_A, y_A, z_A)$ and $M(x, y, z)$, the equation (3.4) becomes

$$\begin{cases} x = x_A + \alpha p_1 + \beta p_2 \\ y = y_A + \alpha q_1 + \beta q_2 \\ z = z_A + \alpha r_1 + \beta r_2 \end{cases}, \qquad \alpha, \beta \in \mathbb{R}, \tag{3.7}$$

and these are the parametric equations of the plane. Again, this system must have an infinity of solutions (α, β), so that

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0, \tag{3.8}$$

which is the analytic equation of the plane determined by a point and two nonparallel directions.

3.1.1 Exercises

1) Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two different points. Prove that the equations of the plane containing P_1 and P_2 and parallel to the vector $\overline{a}(l, m, n)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0.$$

2) Find the equation of the plane containing $P(2, 1, -1)$ and perpendicular on the vector $\overline{n}(1, -2, 3)$.

3) Find the equation of the plane determined by $O(0, 0, 0)$, $P_1(3, -1, 2)$ and $P_2(4, -2, -1)$.

4) Find the equation of the plane containing $P(3, 4, -5)$ and parallel to both $\overline{a}_1(1, -2, 4)$ and $\overline{a}_2(2, 1, 1)$.

5) Find the equation of the plane containing the points $P_1(2, -1, -3)$ and $P_2(3, 1, 2)$ and parallel to the vector $\overline{a}(3, -1, -4)$.

6) Find the equation of the plane passing through $P(7, -5, 1)$ and which determines on the positive half-axes three segments of the same length.

7) Find the equation of the plane containing the perpendicular lines through $P(-2, 3, 5)$ on the planes

$$\pi_1 : 4x + y - 3z + 13 = 0, \quad \pi_2 : x - 2y + z - 11 = 0.$$

8) Find the equation of the plane passing through P and having the normal vector $\overline{n}$:

   a) $P(2, 6, 1)$, $\overline{n}(1, 4, 2)$;
   b) $P(1, 0, 0)$, $\overline{n}(0, 1, 1)$.

9) Find the equation of the plane passing through the given points:

   a) $(-2, 1, 1)$, $(0, 2, 3)$ and $(1, 0, -1)$;
   b) $(3, 2, 1)$, $(2, 1, -1)$ and $(-1, 3, 2)$.

10) Show that the points $(1, 0, -1)$, $(0, 2, 3)$, $(-2, 1, 1)$ and $(4, 2, 3)$ are coplanar.

3.2 Analytic Representations of Lines

As in the case of $\mathcal{E}_2$, a line d in the 3-space is completely determined by a point $P_0(x_0, y_0, z_0)$ of the line and a nonzero vector $\overline{v}(p, q, r)$, parallel to d. In order to represent a line in the 3-dimensional case, the parametric equations are, generally, the most convenient.

If $P(x, y, z)$ is an arbitrary point of the line d, then the vectors $\overline{P_0P}$ and $\overline{v}$ are linearly dependent in V_3 (see Figure 3.3) and there exists $t \in \mathbb{R}$, such that

$$\overline{P_0P} = t\overline{v}. \tag{3.9}$$

Since $\overline{P_0P}(x - x_0, y - y_0, z - z_0)$, by decomposing (3.9) in components, one obtains the *parametric* equations of the line passing through $P_0(x_0, y_0, z_0)$ and parallel to $\overline{v}(a, b, c)$:

$$\begin{cases} x = x_0 + pt \\ y = y_0 + qt \\ z = z_0 + rt \end{cases}, \qquad t \in \mathbb{R}. \tag{3.10}$$

The vector $\overline{v}(p, q, r)$ is called the *director* vector of the line d.

Figure 3.3:

For a fixed point O in the space, the vector $\overline{P_0P}$ can be expressed as the difference $\overline{r} - \overline{r}_0$ (see Figure 3.4) and the equation (3.9) becomes

$$\overline{r} = \overline{r}_0 + t\overline{v}, \qquad t \in \mathbb{R}, \tag{3.11}$$

said to be the *vector* equation of the line in 3-space.

Figure 3.4:

Expressing t three times in (3.10), one obtains the *symmetric* equations of the line d:

$$\frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r}. \tag{3.12}$$

Remark: The director vector $\overline{v}$ is a nonzero vector, i.e. at least one of its components is different from zero. As in the 2-dimensional case, if $p = 0$, for

instance, the meaning of $\dfrac{x-x_0}{0}$ is that $x = x_0$.

A line d can be determined by two different points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ of it. In this case, the director vector of d is

$$\overline{P_1P_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

and the equations of the line determined by two points are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}. \tag{3.13}$$

Given two distinct and nonparallel planes $\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$ and $\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$ (the planes π_1 and π_2 are parallel when their normal vectors $\overline{n}_1(A_1, B_1, C_1)$ and $\overline{n}_2(A_2, B_2, C_2)$ are parallel, i.e. the rank of the matrix $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ is 1), they have an entire line d in common. Then, a line in 3-space can be determined as the intersection of two nonparallel planes:

$$d : \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}, \quad \text{with} \quad \text{rank} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2. \tag{3.14}$$

3.2.1 Exercises

1) Let d_1 and d_2 be two lines in $\mathcal{E}_3$, given by

$$d_1 : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-5}{6} \quad \text{and} \quad d_2 : \frac{x-1}{1} = \frac{y+1}{1} = \frac{z-5}{-3}.$$

   b) Find the parametric equations of d_1 and d_2.

   b) Prove that they are incident and find the coordinates of their intersection point.

   c) Find the equation of the plane determined by d_1 and d_2.

2) Given two lines

$$d_1 : x = 1 + t, y = 1 + 2t, z = 3 + t, \quad t \in \mathbb{R}$$

and

$$d_2 : x = 3 + s, y = 2s, z = -2 + s, \quad s \in \mathbb{R},$$

prove that $d_1 \parallel d_2$ and find the equation of the plane determined by d_1 and d_2.

3) Find the parametric equations of the line

$$\begin{cases} -2x+3y+7z+2=0 \\ x+2y-3z+5=0 \end{cases}.$$

4) Find the parametric equations of the line passing through P_1 and P_2:

   a) $P_1(5,-2,1)$ and $P_2(2,4,2)$;

   b) $P_1(4,0,7)$ and $P_2(-1,-1,2)$

5) Find the parametric equations of the line passing through $(-1,2,4)$ and parallel to $\overline{v}(3,-4,1)$.

6) Find the equations of the line passing through the origin and parallel to the line

$$\begin{cases} x &= & t \\ y &= & -1+t \\ z &= & 2 \end{cases}.$$

3.3 Relative Positions for Lines and Planes

3.3.1 Relative Positions of Two Lines

Let d_1 and d_2 be two lines in $\mathcal{E}_3$, of director vectors $\overline{v}_1(p_1,q_1,r_1) \neq \overline{0}$, respectively $\overline{v}_2(p_2,q_2,r_2) \neq \overline{0}$. The parametric equations of these lines are

$$d_1 : \begin{cases} x = x_1 + p_1 t \\ y = y_1 + q_1 t \\ z = z_1 + r_1 t \end{cases}, \quad t \in \mathbb{R}; \qquad \text{and} \quad d_2 : \begin{cases} x = x_2 + p_2 s \\ y = y_2 + q_2 s \\ z = z_2 + r_2 s \end{cases}, \quad s \in \mathbb{R}.$$

The set of the intersection points of d_1 and d_2 is given by the set of the solutions of the system of equations

$$\begin{cases} x_1 + p_1 t = x_2 + p_2 s \\ y_1 + q_1 t = y_2 + q_2 s \\ z_1 + r_1 t = z_2 + r_2 s \end{cases}. \tag{3.15}$$

- If the system (3.15) has a unique solution (t_0, s_0), then the lines d_1 and d_2 have exactly one intersection point P_0, corresponding to t_0 (or s_0). One says that the lines are *concurrent* (or *incident*); $\{P_0\} = d_1 \cap d_2$. The vectors $\overline{v}_1$ and $\overline{v}_2$ are, necessarily, linearly independent.

Figure 3.5:

- If the system (3.15) has an infinity of solutions, then the two lines have an infinity of points in common, so that the lines coincide. They are *identical*; $d_1 = d_2$. There exists $\alpha \in \mathbb{R}^*$ such that $\overline{v}_1 = \alpha\overline{v}_2$ (their director vectors are linearly dependent) and any arbitrary point of d_1 belongs to d_2 (and reciprocally).

- Suppose that the above system of equations is not compatible. There are two possible situations.

  - If the director vectors are linearly dependent (there exists $\alpha \in \mathbb{R}^*$ such that $\overline{v}_1 = \alpha\overline{v}_2$ or, equivalently, $\frac{p_1}{p_2} = \frac{q_1}{q_2} = \frac{r_1}{r_2}$), then the lines are *parallel*; $d_1 \parallel d_2$.

Figure 3.6:

  - If the director vectors are linearly independent, then one deals with *skew* lines (nonparallel and nonincident); $d_1 \cap d_2 = \emptyset$ and $d_1 \nparallel d_2$.

3.3.2 Relative Positions of Two Planes

Let

$$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0, \qquad \overline{n}_1(a_1, b_1, c_1) \neq \overline{0}$$

and

$$\pi_2 : a_2x + b_2y + c_2z + d_2 = 0, \qquad \overline{n}_2(a_2, b_2, c_2) \neq \overline{0}$$

be two planes, having the normal vectors $\overline{n}_1$, respectively $\overline{n}_2$.

The intersection of these planes is given by the solution of the system of equations

$$\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0 \end{cases} . \tag{3.16}$$

- If rank $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$, then the system (3.16) is compatible and the planes have a line in common. They are *incident*; $\pi_1 \cap \pi_2 = d$.

Figure 3.7:

- If rank $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 1$, then the rows of the matrix are proportional, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, which means that the normal vectors of the planes are linearly dependent. There are two possible situations:

Figure 3.8:

- If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \neq \dfrac{d_1}{d_2}$, then the system (3.16) is not compatible, and the planes are *parallel*; $\pi_1 \parallel \pi_1$.
- If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = \dfrac{d_1}{d_2}$, then the planes are *identical*; $\pi_1 = \pi_2$.

Bundle of Planes

Given a line d, the set of all the planes containing the line d is said to be the *bundle* of planes through d. Let us suppose that d is determined as the intersection of two planes π_1 and π_2, i.e.

$$d : \begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0 \end{cases}, \qquad \text{with rank} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2.$$

The equation of the bundle is

$$\lambda_1\pi_1 + \lambda_2\pi_2 = 0, \qquad (\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{(0,0)\}. \tag{3.17}$$

Remark: Since not both λ_1 and λ_2 are zero, one may suppose that $\lambda_1 \neq 0$ and divide in (3.17) by λ_1; one obtains the *reduced* equation of the bundle:

$$\pi_1 + \lambda\pi_2 = 0,$$

which contains all the planes through d, except π_2.

3.3.3 Relative Positions of a Line and a Plane

Let

$$d : \begin{cases} x = x_1 + pt \\ y = y_1 + qt \\ z = z_1 + rt \end{cases}, \qquad p^2 + q^2 + r^2 > 0$$

be a line of director vector $\overline{v}(p, q, r)$ and

$$\pi : ax + by + cz + d = 0, \qquad a^2 + b^2 + c^2 > 0$$

be a plane of normal vector $\overline{n}(a, b, c)$.

The intersection between d and π is given by the solutions of the equation

$$a(x_1 + pt) + b(y_1 + qt) + c(z_1 + rt) + d = 0. \tag{3.18}$$

- If (3.18) has a unique solution t_0, then d and π have one intersection point P_0, corresponding to the parameter t_0. The line and the plane are *incident*; $d \cap \pi = \{P_0\}$.

Figure 3.9:

- If (3.18) has an infinity of solutions, then d and π have the entire line d in common and d is *contained* into π; $d \subset \pi$. In this case, the normal vector $\overline{n}$ of π is orthogonal on the director vector $\overline{v}$ of d (then $\overline{n} \cdot \overline{v} = 0$, or $ap + bq + cr = 0$) and any point of d is contained into π.

Figure 3.10:

- If (3.18) has no solutions, then the line d is *parallel* to the plane π; $d \parallel \pi$.

Figure 3.11:

3.3.4 Exercises

1) Find the equations of the line passing through $P(6, 4, -2)$ and parallel to the line $d : \dfrac{x}{2} = \dfrac{y-1}{-3} = \dfrac{z-5}{6}$.

2) Given the lines

$$d_1 : x = 4 - 2t, y = 1 + 2t, z = 9 + 3t$$

and

$$d_2 : \frac{x-1}{2} = \frac{y+2}{3} = \frac{z-4}{2},$$

find the intersection points between the two lines and the coordinate planes.

3) Let d_1 and d_2 be the lines given by

$$d_1 : x = 3 + t, y = -2 + t, z = 9 + t, \quad t \in \mathbb{R}$$

and

$$d_2 : x = 1 - 2s, y = 5 + s, z = -2 - 5s, \quad s \in \mathbb{R}.$$

a) Prove they are coplanar.

b) Find the equation of the line passing through the point $P(4, 1, 6)$ and orthogonal on the plane determined by d_1 and d_2.

4) Prove that the intersection lines of the planes $\pi_1 : 2x - y + 3z - 5 = 0$, $\pi_2 : 3x + y + 2z - 1 = 0$ and $\pi_3 : 4x + 3y + z + 2 = 0$ are parallel.

5) Verify that the lines

$$d_1 : \frac{x-3}{1} = \frac{y-8}{3} = \frac{z-3}{4}$$

and

$$d_2 : \frac{x-4}{1} = \frac{y-9}{2} = \frac{z-9}{5}$$

are coplanar and find the equation of the plane determined by the two lines.

6) Determine whether the line

$$\begin{cases} x = 3 + 8t \\ y = 4 + 5t \\ z = -3 - t \end{cases}$$

is parallel to the plane $x - 3y + 5z - 12 = 0$.

7) Find the intersection point between the line

$$\begin{cases} x = 3 + 8t \\ y = 4 + 5t \\ z = -3 - t \end{cases}$$

and the plane $x - 3y + 5z - 12 = 0$.

8) Prove that the lines

$$d_1 : \begin{cases} x = 1 + 4t \\ y = 5 - 4t \\ z = -1 + 5t \end{cases} \quad \text{and} \quad d_2 : \begin{cases} x = 2 + 8t \\ y = 4 - 3t \\ z = 5 + t \end{cases}$$

are skew.

9) Find the parametric equations of the line passing through $(5, 0, -2)$ and parallel to the planes $x - 4y + 2z = 0$ and $2x + 3y - z + 1 = 0$.

10) Find the equation of the plane containing the point $P(2, 0, 3)$ and the line

$$d : \begin{cases} x &= & -1 + t \\ y &= & t \\ z &= & -4 + 2t \end{cases}$$

11) Show that the line

$$\begin{cases} x = 0 \\ y = t \\ z = t \end{cases}$$

is contained into the plane $6x + 4y - 4z = 0$.

12) Let $M_1(2, 1, -1)$ and $M_2(-3, 0, 2)$ be two points. Find:

   a) the equation of the bundle of planes passing through M_1 and M_2;

   b) the plane π from the bundle, which is orthogonal on xOy;

   c) the plane from the bundle, which is orthogonal on π.

13) Given the points $A(1, 2\alpha, \alpha)$, $B(3, 2, 1)$, $C(-\alpha, 0, \alpha)$ and $D((-1, 3, -3)$, find the parameter α, such that the bundle of planes passing through AB has a common point with the bundle of planes passing through CD.

14) Given the planes

$$\pi_1 : 2x + y - 3z - 5 = 0$$

and

$$\pi_2 : x + 3y + 2z + 1 = 0,$$

find the equations of the bisector planes of the dihedral angle and choose which one belongs to the acute dihedral angle.

3.4 Metric Problems Concerning Angles

3.4.1 The Angle Determined by Two Lines

Let d_1 and d_2 be two lines on $\mathcal{E}_3$, whose director vectors are $\overline{v}_1$ respectively $\overline{v}_2$. The *angle* determined by d_1 and d_2 is considered to be the acute or right angle formed by d_1 and d_2. It is denoted by $\widehat{(d_1, d_2)}$.

Theorem 3.4.1.1. *The measure of the angle determined by d_1 and d_2 is given by*

$$m\widehat{(d_1, d_2)} = \begin{cases} m\widehat{(\overline{v}_1, \overline{v}_2)}, & \textit{if } \ \overline{v}_1 \cdot \overline{v}_2 \geq 0 \\ \pi - m\widehat{(\overline{v}_1, \overline{v}_2)}, & \textit{if } \ \overline{v}_1 \cdot \overline{v}_2 < 0 \end{cases} \tag{3.19}$$

Figure 3.12:

Proof: The assertion is immediate. □

Using the definition of the dot product of two vectors, (3.19) becomes

$$m\widehat{(d_1, d_2)} = \begin{cases} \arccos \dfrac{\overline{v}_1 \cdot \overline{v}_2}{|\overline{v}_1||\overline{v}_2|}, & \text{if } \ \overline{v}_1 \cdot \overline{v}_2 \geq 0 \\ \pi - \arccos \dfrac{\overline{v}_1 \cdot \overline{v}_2}{|\overline{v}_1||\overline{v}_2|}, & \text{if } \ \overline{v}_1 \cdot \overline{v}_2 < 0 \end{cases} \tag{3.20}$$

Remark: Two (concurrent or skew) lines d_1 and d_2, having the director vectors $\overline{v}_1(p_1, q_1, r_1)$, respectively $\overline{v}_2(p_2, q_2, r_2)$, are orthogonal if their director vectors are orthogonal.

$$d_1 \perp d_2 \iff \overline{v}_1 \cdot \overline{v}_2 = 0 \iff p_1 p_2 + q_1 q_2 + r_1 r_2 = 0. \tag{3.21}$$

3.4.2 The Angle Determined by a Line and a Plane

Let d be a line of director vector $\overline{v}(p,q,r)$ and π be a plane of normal vector $\overline{n}(a,b,c)$. The *angle* determined by d and π, denoted by $\widehat{(d,\pi)}$, is the angle determined by d and the orthogonal projection d' of d on π.

Figure 3.13:

Theorem 3.4.2.1. *The measure of the angle determined by the line d and the plane π is given by*

$$m\widehat{(d,\pi)} = \begin{cases} \dfrac{\pi}{2} - m\widehat{(\overline{v},\overline{n})}, & \text{if} \quad \overline{v}\cdot\overline{n} \geq 0 \\ m\widehat{(\overline{v},\overline{n})} - \dfrac{\pi}{2}, & \text{if} \quad \overline{v}\cdot\overline{n} < 0 \end{cases} \tag{3.22}$$

Figure 3.14:

The formula (3.22) has the alternative form

$$m\widehat{(d,\pi)} = \begin{cases} \dfrac{\pi}{2} - \arccos\dfrac{\overline{v}\cdot\overline{n}}{|\overline{v}||\overline{n}|}, & \text{if} \quad \overline{v}\cdot\overline{n} \geq 0 \\ \arccos\dfrac{\overline{v}\cdot\overline{n}}{|\overline{v}||\overline{n}|} - \dfrac{\pi}{2}, & \text{if} \quad \overline{v}\cdot\overline{n} < 0 \end{cases} \tag{3.23}$$

Remarks:

1) The line d is parallel to the plane π if the vector $\overline{v}$ is orthogonal to $\overline{n}$, hence
$$d \parallel \pi \iff \overline{v} \cdot \overline{n} = 0 \iff pa + qb + rc = 0. \tag{3.24}$$

2) The line d is orthogonal to the plane π if $\overline{v}$ is parallel to $\overline{n}$. Then
$$d \perp \pi \iff \overline{v} \parallel \overline{n} \iff \exists\, \alpha \in \mathbb{R}^* : \overline{n} = \alpha \overline{d}. \tag{3.25}$$

3.4.3 The Angle Determined by Two Planes

Let π_1 and π_2 be two planes of normal vectors $\overline{n}_1(a_1, b_1, c_1)$, respectively $\overline{n}_2(a_2, b_2, c_2)$. The *angle* determined by π_1 and π_2, denoted by $(\widehat{\pi_1, \pi_2})$, is the acute or right dihedral angle of π_1 and π_2.

Figure 3.15:

Theorem 3.4.3.1. *The measure of the angle determined by π_1 and π_2 is given by*
$$m(\widehat{\pi_1, \pi_2}) = \begin{cases} m(\widehat{\overline{n}_1, \overline{n}_2}), & \textit{if } \ \overline{n}_1 \cdot \overline{n}_2 \geq o \\ \pi - m(\widehat{\overline{n}_1, \overline{n}_2}), & \textit{if } \ \overline{n}_1 \cdot \overline{n}_2 < o \end{cases} \tag{3.26}$$

The formula (3.26) can be written in the form
$$m(\widehat{\pi_1, \pi_2}) = \begin{cases} \arccos \dfrac{\overline{n}_1 \cdot \overline{n}_2}{|\overline{n}_1||\overline{n}_2|}, & \text{if } \ \overline{n}_1 \cdot \overline{n}_2 \geq o \\ \pi - \arccos \dfrac{\overline{n}_1 \cdot \overline{n}_2}{|\overline{n}_1||\overline{n}_2|}, & \text{if } \ \overline{n}_1 \cdot \overline{n}_2 < o \end{cases} \tag{3.27}$$

Remark: The planes π_1 and π_2 are orthogonal if and only if their normal vectors are orthogonal, hence
$$\pi_1 \perp \pi_2 \iff \overline{n}_1 \cdot \overline{n}_2 = 0 \iff a_1 a_2 + b_1 b_2 + c_1 c_2 = 0. \tag{3.28}$$

Figure 3.16:

3.4.4 Exercises

1) Find the angle determined by d_1 and d_2:

   a) $d_1 : x = 4 - t, y = 3 + 2t, z = -2t, t \in \mathbb{R}$
   $d_2 : x = 5 + 2s, y = 1 + 3s, z = 5 - 6s, s \in \mathbb{R}$.

   b) $d_1 : \dfrac{x-1}{2} = \dfrac{y+5}{7} = \dfrac{z-1}{-1}, \quad d_2 : \dfrac{x+3}{-2} = \dfrac{y-9}{1} = \dfrac{z}{4}.$

2) Find the angle determined by the planes

$$\pi_1 : x - \sqrt{2}y + z - 1 = 0 \quad \text{and} \quad \pi_2 : x + \sqrt{2}y - z + 3 = 0.$$

3) Find the equations of the projection of the line

$$d : \begin{cases} 2x - y + z - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

   on the plane $\pi : x + 2y - z = 0$.

4) Find the orthogonal projection of the point $P(2, 1, 1)$ on the plane $\pi : x + y + 3z + 5 = 0$.

5) Find the angle determined by the lines

$$d_1 : \begin{cases} x + 2y + z - 1 = 0 \\ x - 2y + z + 1 = 0 \end{cases} \quad \text{and} \quad d_2 : \begin{cases} x - y - z - 1 = 0 \\ x - y + 2x + 1 = 0 \end{cases}.$$

6) Find the angle determined by the planes

$$\pi_1 : x + 3y + 2z + 1 = 0 \quad \text{and} \quad \pi_2 : 3x + 2y - z - 6 = 0.$$

7) Find the angle determined by the plane xOy and the line M_1M_2, where $M_1(1, 2, 3)$ and $M_2(-2, 1, 4)$.

3.5 Metric Problems Concerning Distances

3.5.1 The Distance From a Point to a Plane

Let $P_0(x_0, y_0, z_0)$ be a point and $\pi : ax+by+cz+d = 0$ (with $a^2+b^2+c^2 > 0$) be a plane in $\mathcal{E}_3$.

Theorem 3.5.1.1. *The distance from the point $P_0(x_0, y_0, z_0)$ to the plane $\pi : ax + by + cz + d = 0$ is given by*

$$d(P_0, \pi) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{3.29}$$

Proof: Let $P_1(x_1, y_1, z_1)$ be an arbitrary point on the plane π and let $\overline{n}(a, b, c)$ be the normal vector of π.

Figure 3.17:

The distance from P_0 to π is the length of the orthogonal projection of the vector $\overline{P_1P_0}$ on the direction of $\overline{n}$ (see Figure 3.17). Then, one has

$$d(P_0, \pi) = |\text{pr}_{\overline{n}}\overline{P_1P_0}| = \frac{|\overline{P_1P_0} \cdot \overline{n}|}{|\overline{n}|} = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} =$$

$$= \frac{|ax_0 + by_0 + cz_0 - (ax_1 + by_1 + cz_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + cy_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.\square$$

3.5.2 The Distance From a Point to a Line

Given a point $P_0(x_0, y_0, z_0)$ and a line $d : \begin{cases} x = x_1 + pt \\ y = y_1 + qt \\ z = z_1 + rt \end{cases}$, $t \in \mathbb{R}$, with $p^2 + q^2 + r^2 > 0$, we present two ways to find the distance from P_0 to d.

I) Let $\overline{v}(p, q, r)$ be the director vector of d and $P_1(x_1, y_1, z_2)$ be an arbitrary point on d. The distance from P_0 to d is the altitude of the parallelogram determined by $\overline{v}$ and $\overline{P_1P_0}$ (see Figure 3.18).

Figure 3.18:

This altitude can be expressed using the area of the parallelogram and one has

$$d(P_0, d) = \frac{|\overline{v} \times \overline{P_1P_0}|}{|\overline{v}|}. \tag{3.30}$$

II) Let π be the plane passing through P_0 and orthogonal on d. Its equation is

$$\pi : p(x - x_0) + q(y - y_0) + r(z - z_0) = 0.$$

Let P_0' be the intersection point of π and d; $\{P_0'\} = d \cap \pi$ (see Figure 3.19). The coordinates of the point P_0' correspond to the parameter t_0, solution of the equation

$$p(x_1 + pt - x_0) + q(y_1 + qt - y_0) + r(z_1 + rt - z_0) = 0.$$

Finally, $d(P_0, d) = d(P_0, P_0')$.

3.5.3 The Distance Between Two Parallel Planes

Let π_1 and π_2 be two parallel planes. Choose an arbitrary point $P_1 \in \pi_1$. Then

$$d(\pi_1, \pi_2) = d(P_1, \pi_2).$$

Figure 3.19:

Figure 3.20:

3.5.4 The Distance Between Two Lines

Let d_1 and d_2 be two lines in the 3-space.

- If the lines are identical or concurrent, then $d(d_1, d_2) = 0$.
- If the lines are parallel, it is enough to choose an arbitrary point $P_1 \in d_1$ and $d(d_1, d_2) = d(P_1, d_2)$.

Figure 3.21:

- If d_1 and d_2 are skew, there exists a unique line which is orthogonal on both d_1 and d_2 and intersects both d_1 and d_2. The length of the segment determined by these intersection points is the distance between the skew lines.

  Suppose that

$$d_1 : \begin{cases} x = x_1 + p_1 t \\ y = y_1 + q_1 t \\ z = z_1 + r_1 t \end{cases}, \quad t \in \mathbb{R} \qquad \text{and} \quad d_2 : \begin{cases} x = x_2 + p_2 s \\ y = y_2 + q_2 s \\ z = z_2 + r_2 s \end{cases}, \quad s \in \mathbb{R}$$

  are, respectively, the parametric equations of the lines of director vectors $\overline{v}_1(p_1, q_1, r_1) \neq \overline{0}$, respectively $\overline{v}_2(p_2, q_2, r_2) \neq \overline{0}$.

  One can determine the equations of two parallel planes $\pi_1 \parallel \pi_2$, such that $d_1 \subset \pi_1$ and $d_2 \subset \pi_2$. The normal vector $\overline{n}$ of these planes has to be orthogonal on both $\overline{v}_1$ and $\overline{v}_2$, hence $\overline{n} = \overline{v}_1 \times \overline{v}_2$.

Figure 3.22:

  Then $\overline{n}(A, B, C)$, with $A = \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}$, $B = \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix}$ and $C = \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$.

  The equations of the planes π_1 and π_2 are:

$$\pi_1 : A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

$$\pi_2 : A(x - x_2) + B(y - y_2) + C(z - z_2) = 0.$$

  Now, the distance between d_1 and d_2 is the distance between the parallel planes π_1 and π_2; $d(d_1, d_2) = d(\pi_1, \pi_2)$, and one has:

  Theorem 3.5.4.1. *The distance between two skew lines d_1 and d_2 is given by*

$$d(d_1, d_2) = \frac{|A(x_1 - x_2) + B(y_1 - y_2) + C(z_1 - z_2)|}{\sqrt{A^2 + B^2 + C^2}}. \tag{3.31}$$

3.5.5 Exercises

1) Find the distance from the point $P(1,2,-1)$ to the line $d: x = y = z$.

2) Find the distance from $(3,1,-1)$ to the plane $22x + 4y - 20z - 45 = 0$.

3) Find the equation of the line passing through $P(4,3,10)$ and orthogonal on the line

$$d: \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{5},$$

the distance from P to d and the coordinates of the symmetrical P' of P with respect to the line d.

4) Find the distance between the lines

$$d_1: \frac{x-1}{2} = \frac{y+1}{3} = \frac{z}{1} \quad \text{and} \quad d_2: \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{3}.$$

5) Find the distance from the point $(0,1,4)$ to the plane

$$\pi: 3x + 6y - 2z - 5 = 0.$$

6) Find the distance between the planes

$$2x - 3y + 4z - 7 = 0 \quad \text{and} \quad 4x - 6y + 8z - 3 = 0.$$

7) Show that the line $\dfrac{x+1}{1} = \dfrac{y-3}{2} = \dfrac{z}{-1}$ and the plane $2x-2y-2z+3=0$ are parallel and find the distance between them.

8) Find the geometric locus of the lines passing through a given point and having a constant distance to a given line.

9) Let $ABCD$ be a tetrahedron and d a line which intersects the faces of the tetrahedron at A', B', C' respectively D'. Prove that the midpoints of the segments $[AA']$, $[BB']$, $[CC']$ and $[DD']$ are coplanar.

10) Let $VABC$ be a regular quadrilateral pyramid of vertex V, having the sides $AB = a$ and $VA = a\sqrt{2}$. Let M be the midpoint of $[VA]$. The plane passing through M and orthogonal on VC determines a section in the pyramid. Find the perimeter and the area of this section.

11) Let $ABCD$ be a cube with side of length a, M and P be the midpoints of $[BC]$ respectively $[AA']$, O the center of the cube, O' the center of the face $A'B'C'D'$ and S the midpoint of $[OO']$. Find the area of the section determined by the plane (MPS) in the cube.

12) Let $ABCD$ be a cube with side of length a, M and P be the midpoints of $[BC]$ respectively $[AA']$ and O' the center of the face $A'B'C'D'$. Find the perimeter and the area of the section determined by the plane (MPO') in the cube.

Chapter 4

Plane Isometries

4.1 General Properties

The Euclidean plane $\mathcal{E}_2$ can be identified with the metric space $(\mathbb{R}^2, d_2)$, where the metric d_2 is the Euclidean metric

$$d_2(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}, \quad \forall\, A(x_A, y_A), B(x_B, y_B).$$

The map $f : \mathcal{E}_2 \to \mathcal{E}_2$ is said to be an *isometry* of the plane $\mathcal{E}_2$ if f conserves the distances, i.e.

$$|f(A)f(B)| = |AB|, \quad \forall\, A, B \in \mathcal{E}_2.$$

(One denotes $|AB| = d_2(A,B)$).

Proposition 4.1.1. *The image of a segment through an isometry $f : \mathcal{E}_2 \to \mathcal{E}_2$ is a segment of the same length.*

Proof: Let A and B be two points on $\mathcal{E}_2$. It is enough to prove that $f([AB]) = [f(A)f(B)]$.

Let $M \in [AB]$. Then $|AM| + |MB| = |AB|$. Since f is an isometry,

$$|f(A)f(M)| + |f(M)f(B)| = |f(A)f(B)|,$$

which means that $f(M) \in [f(A)f(B)]$. Then $f([AB]) \subseteq [f(A)f(B)])$.

Conversely, take $Y \in [f(A)f(B)]$. Since

$$|f(A)Y| \leq |f(A)f(B)| \,\text{and}\, |f(A)f(B)| = |AB|,$$

there exists $X \in [AB]$ such that $|AX| = |f(A)Y|$. The points X and Y have the following properties:

$$\begin{cases} |AX| = |f(A)f(X)| = |f(A)Y| \\ X \in [AB] \Longrightarrow f(X) \in [f(A)f(B)] \end{cases}$$

Then $f(X) = Y$ and $Y \in f([AB])$, such that $[f(A)f(B)] \subseteq f([AB])$. $\square$

Proposition 4.1.2. *Let $f : \mathcal{E}_2 \to \mathcal{E}_2$ be an isometry. Then:*

1) The image of a half-line is a half-line;

2) The image of a line is a line;

3) If A, B and C are three noncollinear points on $\mathcal{E}_2$, then so are their images $f(A)$, $f(B)$ and $f(C)$;

4) The image of a triangle ΔABC is triangle $\Delta f(A)f(B)f(C)$, such that

$$\Delta ABC \equiv \Delta f(A)f(B)f(C);$$

5) The image of an angle $\widehat{AOB}$ is an angle $f(A)\widehat{f(O)}f(B)$ having the same measure;

6) Two orthogonal lines are transformed into two orthogonal lines;

7) Two parallel lines are transformed into two parallel lines.

Proposition 4.1.3. *Any isometry $f : \mathcal{E}_2 \to \mathcal{E}_2$ is a surjective map.*

Proof: Choose three noncollinear points A, B and C and let A', B' and C' be their images (see Figure 4.1).

Let $X' \in \mathcal{E}_2$. Let $M' \in A'B'$ and $N' \in A'C'$, such that $X'M' \parallel A'C'$ and $X'N' \parallel A'B'$. Since the image of a line through f is a line, there exist $M \in AB$ and $N \in AC$, such that $f(M) = M'$ and $f(N) = N'$. Construct the parallelogram $AMXN$. Since $f(M) = M'$ and $f(N) = N'$, then $f(MX) = M'X'$ (the image of the line passing through M and parallel to AC is the line passing through $f(M) = M'$ and parallel to $A'C'$) and $f(NX) = N'X'$, therefore $f(X) = X'$ and f is surjective. $\square$

Denote the set of isometries of the plane by $\mathrm{Iso}(\mathcal{E}_2)$;

$$\mathrm{Iso}(\mathcal{E}_2) = \{f : \mathcal{E}_2 \to \mathcal{E}_2, f \text{ isometry}\}.$$

Figure 4.1:

Proposition 4.1.4. (Iso $(\mathcal{E}_2), \circ)$ *is a group, called the* group of isometries *of the plane.*

Proof: $(\mathrm{Iso}(\mathcal{E}_2), \circ)$ is a subgroup of the group of bijective maps from $\mathcal{E}_2$ to $\mathcal{E}_2$,

$$S(\mathcal{E}_2) = \{f : \mathcal{E}_2 \to \mathcal{E}_2, f \,\text{bijection}\}.$$

Indeed, it have been proved that any isometry is surjective. Now, given two points A and B such that $f(A) = f(B)$, it follows that $|f(A)f(B)| = 0$, so $|AB| = 0$ and $A = B$, hence f is injective. Then $\mathrm{Iso}(\mathcal{E}_2) \subset S(\mathcal{E}_2)$.

Moreover, if f and g are isometries, then

$$|f \circ g^{-1}(A) f \circ g^{-1}(B)| = |f(g^{-1}(A)) f(g^{-1}(B))| =$$

$$= |g^{-1}(A) g^{-1}(B)| = |g(g^{-1}(A)) g(g^{-1}(B))| = |AB|,$$

hence $f \circ g^{-1}$ is an isometry. □

- A point $A \in \mathcal{E}_2$ is a *fixed point* for the isometry f if $f(A) = A$;
- A line $d \in \mathcal{E}_2$ is said to be *invariant* with respect to f if $f(d) = d$ (obviously, a line whose points are all fixed is invariant, while the converse is not necessarily true).

Proposition 4.1.5. *1) If A and B are two fixed points for the isometry f, then any point of the line AB is fixed for f;*

2) If the isometry f has three noncollinear fixed points, then f is the identity of the plane, $f = 1_{\mathcal{E}_2}$.

Proof: 1) Let C be an arbitrary point of the line AB and suppose, for instance, that $C \in [AB]$. Then, its image $f(C) \in [f(A)f(B)] = [AB]$. Moreover, $|AC| + |CB| = |AB|$, hence $|Af(C)| + |f(C)B| = |AB|$. It follows necessarily that $f(C) = C$, therefore C is a fixed point for f.

2) Let A, B and C be three noncollinear fixed points for f and choose an arbitrary point $X \in \mathcal{E}_2$. Let $M \in AB$ and $N \in CA$, such that $XM \parallel AC$ and $XN \parallel AB$ (see Figure 4.2).

Figure 4.2:

The points M and N are fixed for f, since they belong to lines determined by two fixed points. Then, the line MN has only fixed points. Hence, the midpoint O of the segment $[MN]$ is a fixed point, so that the line AO contains only fixed points. Therefore, X is a fixed point for f. $\square$

4.2 Symmetries

Let d be a line in $\mathcal{E}_2$. The map $s_d : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$s_d(P) = P', \quad \text{where } P' \text{ is the symmetrical of } P \text{ with respect to the line } d,$$

is called *axial symmetry*. The line d is the *axis* of the symmetry.

Let be given a point O in the plane. The map $s_O : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$s_O(P) = P', \quad \text{where } P' \text{ is the symmetrical of } P \text{ with respect to the point } P,$$

is called *central symmetry*. The point O is the *center* of the symmetry.

A map $f : \mathcal{E}_2 \to \mathcal{E}_2$, having the property that $f \circ f = 1_{\mathcal{E}_2}$, is an *involution*.

Theorem 4.2.1. *The axial symmetries and the central symmetries are involutive isometries.*

Figure 4.3:

Figure 4.4:

Proof: Let s_O be a central symmetry. It os obvious that $s_O \circ s_O = 1_{\mathcal{E}_2}$, hence s_O is an involution. Moreover, s_O is a bijection and $s_O^{-1} = s_O$.

Let P and Q be two arbitrary points on $\mathcal{E}_2$ and let P' and Q' be their symmetricals with respect to O.

Figure 4.5:

The triangles ΔPOQ and $\Delta P'OQ'$ are congruent, hence $|PQ| = |P'Q'|$ and s_O is an isometry.

Similarly, an axial symmetry s_d is an isometry with $s_d^{-1} = s_d$ and an involution. □

Remarks:

1) Any point lying on the axis d of an axial symmetry s_d is a fixed point for s_d;
2) Any line d' which is orthogonal on the axis d of an axial symmetry s_d is invariant with respect to s_d (i.e. $s_d(d') = d'$);
3) The center O of a central symmetry s_O is fixed for s_O;
4) Any line passing through the center O of a central symmetry s_O is invariant with respect to s_O.

We saw in Theorem 4.2.1 that the symmetries are involutions. The following result shows that, excepting the identity of $\mathcal{E}_2$, the symmetries are the only involutive isometries.

Proposition 4.2.2. *Let $f \in \text{Iso}\ (\mathcal{E}_2)$ be an involutive isometry. Then f is either an axial symmetry, or a central symmetry, or $1_{\mathcal{E}_2}$.*

Proof: Any involutive isometry has at least a fixed point. Indeed, if A is an arbitrary point on the plane and $A' = f(A)$, then

$$f(A') = f(f(A)) = (f \circ f)(A) = A.$$

The image, through f, of the midpoint M of the segment $[AA']$ is the midpoint of the segment $[A'A]$, hence M. Then M is a fixed point for f.

Suppose that f has a unique fixed point O. This point must be the midpoint of any segment $[AA']$ in the plane, so that for any $A \in \mathcal{E}_2$, one has $A' = s_O(A)$. Then, $f = s_O$.

If f has at least two fixed points A and B, then any point of the line $d = AB$ is fixed (see Proposition 4.1.5). If f has another fixed point P, not on d, then $f = 1_{\mathcal{E}_2}$.

Suppose that all the fixed points of f are situated on d. Take an arbitrary point $M \in \mathcal{E}_2 \setminus \{d\}$ and let M_0 be the orthogonal projection of M on d. Then M_0 is a fixed point for f (since it lies on d) and the line MM_0 is invariant (since it is orthogonal on d). The image M' of M will be on MM_0 and $|MM_0| = |M'M_0|$ (f is an isometry). Moreover, $M' \neq M$ (since $M \notin d$, hence M is not a fixed point), and then $M' = s_d(M)$ and $f = s_d$. □

Theorem 4.2.3. *Given A, B, C, A', B', $C' \in \mathcal{E}_2$, such that A, B and C are noncollinear and $\Delta ABC \equiv \Delta A'B'C'$, there exists a unique isometry $f \in \text{Iso}(\mathcal{E}_2)$, with*

$$f(A) = A', f(B) = B', f(C) = C'.$$

Proof: It is clear that, if $\Delta ABC \neq \Delta A'B'C'$, an isometry having the required properties does not exists.

We prove first that, if such an isometry does exist, then it is unique. Suppose there are two isometries $f, g \in \text{Iso}\,(\mathcal{E}_2)$, such that $f(A) = A'$, $f(B) = B'$, $f(C) = C'$, and $g(A) = A'$, $g(B) = B'$, $g(C) = C'$. The map $h = f^{-1} \circ g$ is also an isometry and $h(A) = A$, $h(B) = B$, $h(C) = C$. Hence h has three noncollinear fixed points, and therefore $h = 1_{\mathcal{E}_2}$ and $f = g$.

Let us prove now the existence of f.

1) If $A = A'$, $B = B'$ and $C = C'$, one can take $f = 1_{\mathcal{E}_2}$;

2) If $A = A'$, $B = B'$ and $C \neq C'$, one can take $f = s_{AB}$;

3) If $A = A'$, $B \neq B'$ and $C \neq C'$, let b be the perpendicular line on the midpoint of the segment $[BB']$ (which passes through the point A, since $|AB| = |AB'|$). Then $B' = s_b(B)$. The points A and B are fixed points for the isometry $s_b \circ f$. If $(s_b \circ f)(C) = C$, then $s_b \circ f$ has three noncollinear fixed points, $s_b \circ f = 1_{\mathcal{E}_2}$ and $f = s_b$. If $(s_b \circ f)(C) \neq C$, then $s_b \circ f = s_{AB}$, and $f = s_b \circ s_{AB}$.

4) If $A \neq A'$, $B \neq B'$ and $C \neq C'$, let a be the perpendicular line on the midpoint of the segment $[AA']$. It is easy to see that $(s_a \circ f)(A) = A$ and, with the same argument as above, the isometry is given either by $f = s_a \circ s_b$, or by $f = s_a \circ s_b \circ s_{AB}$. $\square$

Theorem 4.2.4. *Given A, B, A', $B' \in \mathcal{E}_2$ (with A and B distinct), such that $[AB] \equiv [A'B']$, there exist exactly two isometries of $\mathcal{E}_2$, with*

$$f(A) = A' \quad \textit{and} \quad f(B) = B'.$$

Proof: Let C be an arbitrary point of the plane $\mathcal{E}_2$, not situated on the line AB. There exist exactly two points C'_1 and C'_2 in the plane, such that $\Delta ABC \equiv \Delta A'B'C'_1$, respectively $\Delta ABC \equiv \Delta A'B'C'_2$ (the points C'_1 and C'_2 are symmetrical with respect to the line $A'B'$). Now, Theorem 4.2.3 assures the existence of a unique isometry f, such that

$$f(A) = A', f(B) = B', f(C) = C'_1$$

and a unique isometry g, such that

$$g(A) = A', g(B) = B', g(C) = C'_2. \quad \square$$

Theorem 4.2.5. *Any isometry is the product of at the most three axial symmetries.*

Proof: Let A, B and C be three noncollinear points, $f \in \text{Iso}\ (\mathcal{E}_2)$ and $A' = f(A)$, $B' = f(B)$, $C' = f(C)$. Theorem 4.2.3 assures that the isometry f is unique and, following the way in which f was constructed before, it can be decomposed into at the most three axial symmetries. □

Corollary 4.2.6. *The axial symmetries generate the group of isometries.*

4.2.1 Exercises

1) Let d be a line and A and B be two points on the plane. Determine the position of a point M on d, such that the sum $|AM|+|MB|$ is minimum.

2) Let C be a point inside the angle $\widehat{AOB}$. Determine two points $P \in (OA$ and $Q \in (OB$, such that the perimeter of the triangle ΔCPQ is minimum.

3) Inscribe in the acute triangle ΔABC a triangle of minimal perimeter.

4) Let $\widehat{AOB}$ be an angle and O be a point inside the triangle. A variable line d, passing through O, cuts AB at M and BC at N. Prove that the area of the triangle ΔMBN is minimal if and only if O is the midpoint of the segment $[MN]$.

5) Let $ABCD$ be a parallelogram, ΔABE and ΔCDF two equilateral triangles situated outside the parallelogram, and let G and H be, respectively, the centers of two squares constructed on AD, respectively BC, outside the parallelogram. Prove that $EGFH$ is a parallelogram.

6) Let $ABCD$ be a parallelogram and O_1, O_2 be the incenters of the triangles ΔABC, respectively ΔADC. Show that AO_1CO_2 is a parallelogram.

7) Let ΔABC be a triangle and H its orthocenter. Prove that the symmetricals of H with respect to AB, BC and CA are situated on the circumscribed circle of the triangle. Also, the symmetricals of H with respect to the midpoints of $[AB]$, $[BC]$ and $[CA]$ are on the circumscribed circle.

4.3 Translations

Let $\overline{v}$ be a vector in V_2. The map $t_{\overline{v}} : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$t_{\overline{v}}(M) = M', \qquad \text{where} \quad \overline{MM'} = \overline{v},$$

is called *translation* of vector $\overline{v}$.

Figure 4.6:

- It is easy to see that $t_{\overline{v}}^{-1} = t_{-\overline{v}}$, therefore

$$t_{\overline{v}}^{-1} = t_{\overline{v}} \iff \overline{v} = \overline{0}.$$

  Hence, the unique involutive translation is the identity of the plane (translation of vector $\overline{0}$);

- Any line, having the direction parallel to the direction of $\overline{v}$, is invariant with respect to $t_{\overline{v}}$.

Theorem 4.3.1. *A product of two central symmetries is a translation. Conversely, any translation can be decomposed into a product of two central symmetries.*

Proof: Let s_A and s_B be two central symmetries.

- If $A = B$, then $s_A \circ s_B = s_A^2 = 1_{\mathcal{E}_2} = t_{\overline{0}}$;

- If $A \neq B$, let M be an arbitrary point on $\mathcal{E}_2$, $M' = s_A(M)$ and

$$M'' = s_B(M') = s_B(s_A(M)) = (s_B \circ s_A)(M).$$

Since $[AB]$ is a midline of the triangle $\Delta MM'M''$, then $\overline{MM''} = 2\overline{AB}$. In conclusion, $s_B \circ s_a = t_{2\overline{AB}}$ (see Figure 4.7).

Conversely, given the translation $t_{\overline{v}}$, one can write $t_{\overline{v}} = s_B \circ s_A$, where A and B are two points on the plane such that $\overline{AB} = \dfrac{1}{2}\overline{v}$ (of course, either A, or B, may be chosen arbitrarily). $\square$

Remark: Since a translation is a product of two central symmetries, then the translation is an isometry.

Figure 4.7:

Figure 4.8:

Proposition 4.3.2. *A product of two axial symmetries, whose axes are parallel, is a translation. Conversely, any translation can be decomposed into a product of two axial symmetries, having parallel axes.*

Proof:

Let d and d' be two parallel lines. Let M be an arbitrary point on the plane, $M' = s_d(M)$ and $M'' = s_{d'}(M') = (s_{d'} \circ s_d)(M)$ (see Figure 4.8). If $\{A\} = MM' \cap d$ and $\{B\} = M'M'' \cap d'$, then $\overline{MM''} = 2\overline{AB}$. One has $s_{d'} \circ s_d = t_{\overline{AB}}$, where the vector $\overline{AB}$ is orthogonal on d and its length is a half of the distance between the lines d and d'.

Conversely, given a translation $t_{\overline{v}}$, one can choose an arbitrary line d, orthogonal on the direction of $\overline{v}$. Let A be an arbitrary point on d and there exists a unique point B, such that $\overline{AB} = \dfrac{1}{2}\overline{v}$. Let d' be the line passing through B and parallel to d. Then $t_{\overline{v}} = s_{d'} \circ s_d$. $\square$

Proposition 4.3.3. *Given two points A and A' in $\mathcal{E}_2$, there exists a unique translation $t_{\overline{v}}$, such that $t_{\overline{v}}(A) = A'$. In particular, if a translation $t_{\overline{v}}$ has a fixed point, then $t_{\overline{v}} = 1_{\mathcal{E}_2}$.*

Proof: $t_{\overline{AA'}}$ is the unique translation having the required property. If $t_{\overline{v}}(P) = P$, then $\overline{v} = \overline{PP} = \overline{0}$, and $t_{\overline{v}} = t_{\overline{0}} = 1_{\mathcal{E}_2}$. $\square$

Denote by

$$T(\mathcal{E}_2) = \{t_{\overline{v}} : \mathcal{E}_2 \to \mathcal{E}_2 : t_{\overline{v}} \text{ translation } \},$$

the set of all the translations of the plane.

Theorem 4.3.4. *$(T(\mathcal{E}_2), \circ)$ is a commutative subgroup of $(\mathrm{Iso}(\mathcal{E}_2), \circ)$, isomorphic to $(V_2, +)$.*

Proof: It is already settled that $T(\mathcal{E}_2) \subset \mathrm{Iso}(\mathcal{E}_2)$. If $t_{\overline{v}}$ and $t_{\overline{w}}$ are two translations, it easy to see that

$$t_{\overline{v}} + t_{\overline{w}} = t_{\overline{v}+\overline{w}} \qquad \text{and} \quad t_{\overline{v}}^{-1} = t_{-\overline{v}}, \tag{4.1}$$

so that the required isomorphism is natural: $f(t_{\overline{v}}) = \overline{v}$. $\square$

Theorem 4.3.5. *Let a, b and c be three parallel lines on $\mathcal{E}_2$. Then $s_a \circ s_b \circ s_c$ is an axial symmetry, whose axis is parallel to the three given lines.*

Proof: The product $s_b \circ s_c$ is a translation (see Proposition 4.3.2). This translation is the product of two axial symmetries, and one may choose one of them to be s_a and the other to be s_d, where d is parallel to a. Then

$$s_a \circ s_b \circ s_c = s_a \circ (s_b \circ s_c) = s_a \circ (s_a \circ s_d) = (s_a \circ s_a) \circ s_d = s_d. \quad \square$$

4.3.1 Exercises

1) Let $\mathcal{C}(O, R)$ be a circle and $A, B \in \mathcal{C}(O, R)$ two fixed points. Find the geometric locus of the orthocenter of the triangle ΔABC, when C is a mobile point on $\mathcal{C}(O, R)$.

2) Let d be a line, A and B be two points situated on the same half-plane with respect to d and $a \geq 0$ a given number. Determine two points M and N on the line d, such that:

   a) $|MN| = a$;

   b) $|AM| + |MN| + |NB|$ is minimum.

3) Construct a trapezium, knowing the lengths of its sides.

4) Let $\mathcal{C}_1(O_1, r_1)$ and $\mathcal{C}_2(O_2, r_2)$ be two circles and d be an arbitrary line. Construct a line d', parallel to d, such that the chords determined by d' on the two circles are of the same length.

5) Let $\mathcal{C}_1(O_1, r_1)$ and $\mathcal{C}_2(O_2, r_2)$ be two circles and $[MN]$ a segment. Construct a segment $[AB]$, parallel to $[MN]$ and of the same length, such that $A \in \mathcal{C}_1(O_1, r_1)$ and $B \in \mathcal{C}_2(O_2, r_2)$.

6) Prove that the group $(T(\mathcal{E}_2), \circ)$ is isomorphic to the additive group of the complex numbers.

4.4 Rotations

An angle $\widehat{AOB}$ is said to be *oriented* if the pair of half-lines $\{[OA, [OB\}$ is ordered. The angle $\widehat{AOB}$ is *positively oriented* if $[OA$ gets over $[OB$ counterclockwisely. Otherwise, $\widehat{AOB}$ is *negatively oriented.* If the measure of the *nonoriented* angle $\widehat{AOB}$ is θ, then the measure of the oriented angle $\widehat{AOB}$ is either θ, or $-\theta$, depending on the orientation of $\widehat{AOB}$.

Figure 4.9:

Let $O \in \mathcal{E}_2$ be a point and $\theta \in [-2\pi, 2\pi]$ be a number. The map $r_{O,\theta} : \mathcal{E}_2 \to \mathcal{E}_2$, given by

$$r_{O,\theta}(M) = M', \quad \text{where} \quad \begin{cases} |OM| = |OM'| \\ m(\widehat{MOM'}) = \theta \end{cases},$$

is called *rotation* of center O and oriented angle θ.

The point O is the *center* of the rotation and the number θ is the *angle* of rotation.

Remarks:

Figure 4.10:

- If $\theta \in \{-2\pi, 0, 2\pi\}$, then the rotation of angle θ is the identity of the plane, $r_{O,-2\pi} = 1_{\mathcal{E}_2}$, $r_{O,0} = 1_{\mathcal{E}_2}$, $r_{O,2\pi} = 1_{\mathcal{E}_2}$;
- If $\theta \in \{-\pi, \pi\}$, then the rotation of angle θ is the central symmetry, $r_{O,-\pi} = s_O$, $r_{O,\pi} = s_O$;
- The rotation $r_{O,\theta}$ is an involution if and only if $\theta \in \{-2\pi, -\pi, 0, \pi, 2\pi\}$;
- The center O of the rotation $r_{O,\theta}$ (of an angle $\theta \neq -2\pi, 0, 2\pi$) is its unique fixed point and there are no invariant lines with respect to $r_{O,\theta}$.

Theorem 4.4.1. *Any rotation is an isometry of $\mathcal{E}_2$.*

Proof: Let $r_{O,\theta}$ be a rotation of center O, let A and B be two arbitrary points in $\mathcal{E}_2$ and $A' = r_{O,\theta}(A)$, $B' = r_{O,\theta}(B)$. Using the definition of the

Figure 4.11:

rotation, one has that $[OA] \equiv [OA']$, $[OB] \equiv [OB']$ and $\widehat{AOB} \equiv \widehat{A'OB'}$. Therefore, $\Delta AOB \equiv \Delta A'OB'$ so that $|AB| = |A'B'|$. $\square$

Theorem 4.4.2. *Any isometry with a unique fixed point O is a rotation of center O.*

Proof: Let $f \in \mathrm{Iso}(\mathcal{E}_2)$ and $f(O) = O$. Let $M \in \mathcal{E}_2 \setminus \{O\}$ be an arbitrary point and $M' = f(M)$, $M' \neq M$. There exist exactly two isometries such that $f(O) = O$ and $f(M) = M'$ (see Theorem 4.2.4) and, since the rotation of center O and angle $\widehat{MOM'}$ and the axial symmetry with respect to the internal bisector of the angle $\widehat{MOM'}$ have these properties, and since the latter has an entire line of fixed points, then f must be a rotation of center O. □

Remarks:

- Sometimes, the Theorem 4.4.2 is given as definition of rotation: the rotation is the isometry having a unique fixed point, called center, or the identity $1_{\mathcal{E}_2}$;
- Let us denote by $R(O)$ the set of all the rotations of center O and angle $\theta \in [-2\pi, 2\pi]$. Then $(R(O), \circ)$ is a commutative subgroup of $(\mathrm{Iso}(\mathcal{E}_2), \circ)$. Indeed, $r_{O,\theta_1} \circ r_{O,\theta_2} = r_{O,\theta_1+\theta_2 (\mod 2\pi)}$ and $r^{-1}_{O,\theta} = r_{O,-\theta}$.

Proposition 4.4.3. *A product of two axial symmetries, whose axis are concurrent, is a rotation. Conversely, any rotation can be expressed as a product of two axial symmetries, whose axis are concurrent at the center of rotation.*

Proof: Let s_a and s_b be two axial symmetries, with $a \cap b = \{O\}$. We shall prove that $s_b \circ s_a$ is a rotation of center O, by verifying that O is the unique fixed point of $s_b \circ s_a$. First,

$$(s_b \circ s_a)(O) = s_b(s_a(O)) = s_b(O) = O,$$

hence O is a fixed point.

Let M be another fixed point of $s_b \circ s_a$. Since $(s_b \circ s_a)(M) = M$, then $(s_b^{-1} \circ s_b \circ s_a)(M) = s_b^{-1}(M)$ and, because the symmetry s_b is involutive, it follows that $s_a(M) = s_b(M) = N$. If the points M and N are distinct, $M \neq N$, then the lines a and b are both perpendicular on the midpoint of the segment $[MN]$, contradiction to the fact that $a \cap b = \{O\}$. Hence $M = N$. Then $M = s_a(M)$ and $M = s_b(M)$, meaning that $M \in a$ and $M \in b$. It follows, necessarily, that $M = O$, and O is the unique fixed point of $s_b \circ s_a$.

Conversely, denote $r = r_{O,\theta}$ and let $A \neq O$ be an arbitrary point and $A' = r(A)$. Let b be the bisector of the angle $\widehat{AOA'}$. The map $s_b \circ r$ is an isometry and, moreover,

$$(s_b \circ r)(O) = s_b(r(O)) = s_b(O) = O$$

and

$$(s_b \circ r)(A) = s_b(r(A)) = s_b(A') = A,$$

so that $s_b \circ r$ has at least two fixed points, A and O. Therefore, any point of OA is fixed and $s_b \circ r$ is either the identity, or the axial symmetry with respect to $a = OA$ (see Theorem 4.2.3). If $s_b \circ r = 1_{\mathcal{E}_2}$, then $s_b = r$, which is impossible, since r has a unique fixed point and s_b has an infinity of fixed points. Then, $s_b \circ r = s_a$, and $r = s_b \circ s_a$.

Because the point A was chosen arbitrarily, one of the axis a and b can be chosen arbitrarily. □

Remark: It is easy to see, in the previous theorem, that the angle of rotation is twice the (oriented) angle $\widehat{aOb}$. Then, the product of two axial symmetries, having orthogonal axes $a \perp b$, is a central symmetry of center O, with $\{O\} = a \cap b$.

Theorem 4.4.4. *Let a, b and c be three lines on $\mathcal{E}_2$, such that $a \cap b \cap c = \{O\}$. Then $s_a \circ s_b \circ s_c$ is an axial symmetry, whose axis d passes through O.*

Proof: Use the Proposition 4.4.3. The product $s_b \circ s_c$ is a rotation of center O. This rotation can be decomposed into a product of two axial symmetries, the axes passing through O, and one of them can be chosen arbitrarily, so that it can be a. Then $s_b \circ s_c = s_a \circ s_d$ and

$$s_a \circ s_b \circ s_c = s_a \circ s_a \circ s_d = s_d. \quad \square$$

4.4.1 Exercises

1) Let ΔABC be an equilateral triangle and M a point in the same plane.

   a) Prove that, if $M \notin \mathcal{C}(A, B, C)$, then there exists a triangle having the sides of lengths $|MA|$, $|MB|$, $|MC|$.

   b) Prove that, if $M \in \mathcal{C}(A, B, C)$, then the length of one of the segments $[MA]$, $[MB]$, $[MC]$ is the sum of the lengths of the two others.

2) Let ΔABC and ΔDBC be two equilateral triangles, symmetrical with respect to BC, and M a point on the circle of center D and radius $|DB|$. If a, b, c are the lengths of the segments $[BC]$, $[CA]$, respectively $[AB]$, prove that $a^2 = b^2 + c^2$.

3) Let a, b and c be three parallel lines. Construct an equilateral triangle ΔABC, such that $A \in a$, $B \in b$, $C \in c$.

4) Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be three circles, of radius $r_1 < r_2 < r_3$, respectively. Determine an equilateral triangle ΔABC, such that $A \in a$, $B \in b$, $C \in c$.

5) Determine a point M in the plane of the triangle ΔABC, such that $|MA| + |MB| + |MC|$ is minimal.

6) Let d be a line and $A \notin d$. Find the geometric locus of the vertex C of an equilateral triangle ΔABC, when $B \in d$.

7) Prove that the product of two rotations is commutative if and only if the centers of the rotations coincide.

4.5 Analytic Form of Isometries

In this paragraph, we determine the equations of the plane isometries. Let xOy be a Cartesian system of coordinates, associated to $\mathcal{E}_2$.

Proposition 4.5.1. *Let $P(x_0, y_0)$ be the center of the central symmetry s_P. The equations of s_P are*

$$\begin{cases} x' = 2x_0 - x \\ y' = 2y_0 - y \end{cases}. \tag{4.2}$$

Proof: Let $M(x, y)$ be an arbitrary point on $\mathcal{E}_2$ and $M' = s_P(M)$ its symmetrical with respect to P, $M' = (x', y')$.

Figure 4.12:

Since P is the midpoint of the segment $[MM']$, then $x_0 = \dfrac{x + x'}{2}$ and $y_0 = \dfrac{y + y'}{2}$, and the equations (4.2) hold. $\square$

Remark: If the center of symmetry is the origin $O(0,0)$ of the system of coordinates, then the equations (4.2) become

$$\begin{cases} x' = -x \\ y' = -y \end{cases}. \tag{4.3}$$

Proposition 4.5.2. *Let $d : ax + by + c = 0$, $a^2 + b^2 > 0$, be a line in $\mathcal{E}_2$. The equations of the axial symmetry s_d are:*

$$\begin{cases} x' = \dfrac{b^2 - a^2}{a^2 + b^2}x - \dfrac{2ab}{a^2 + b^2}y - \dfrac{2ac}{a^2 + b^2} \\ y' = -\dfrac{2ab}{a^2 + b^2}x - \dfrac{b^2 - a^2}{a^2 + b^2}y - \dfrac{2bc}{a^2 + b^2} \end{cases}. \tag{4.4}$$

Proof: One may suppose that $b \neq 0$. Let $M(x, y)$ be an arbitrary point and $M' = s_d(M)$, $M'(x', y')$.

Figure 4.13:

The points M and M' are symmetrical with respect to d if and only if the line passing through M and M' is orthogonal on d and the midpoint P of the segment $[MM']$ belongs to d.

The equation of the line determined by M and M' is $\dfrac{X - x}{x' - x} = \dfrac{Y - y}{y' - y}$. The orthogonality condition gives $a(y' - y) = b(x' - x)$.

The midpoint of $[MM']$ is a point of d if and only if

$$a\left(\frac{x + x'}{2}\right) + b\left(\frac{y + y'}{2}\right) + c = 0.$$

Then, the coordinates (x', y') of M' are the solution of the system of equation

$$\begin{cases} ax' + by' & = & -(ax + by + 2c) \\ bx' - ay' & = & bx - ay \end{cases}$$

and one obtains (4.4). $\square$

Remarks:

- If the line d passes through the origin O, then $c = 0$ and (4.4) becomes

$$\begin{cases} x' = \dfrac{b^2 - a^2}{a^2 + b^2}x - \dfrac{2ab}{a^2 + b^2}y \\ y' = -\dfrac{2ab}{a^2 + b^2}x - \dfrac{b^2 - a^2}{a^2 + b^2}y \end{cases} . \tag{4.5}$$

- If the line d is parallel to Ox, then $a = 0$ and (4.4) turns into

$$\begin{cases} x' = x \\ y' = -y - \dfrac{2c}{b} \end{cases} . \tag{4.6}$$

- If the line d is parallel to Oy, then $b = 0$ and (4.4) converts into

$$\begin{cases} x' = -x - \dfrac{2c}{a} \\ y' = y \end{cases} . \tag{4.7}$$

Proposition 4.5.3. *Let $\overline{v}(x_0, y_0)$ be a vector. The equations of the translation $t_{\overline{v}}$ of vector $\overline{v}$ are*

$$\begin{cases} x' = x + x_0 \\ y' = y + y_0 \end{cases} . \tag{4.8}$$

Proof:

Figure 4.14:

If $M(x, y)$ is an arbitrary point and $M' = t_{\overline{v}}(M)$, $M'(x', y')$, then $\overline{MM'} = \overline{v}$. By identifying the components of the vectors, one obtains the equations (4.8). $\square$

Proposition 4.5.4. *Let $f \in \mathrm{Iso}(\mathcal{E}_2)$ be an isometry, having the origin $O(0,0)$ as fixed point. The equations of f are*

$$\begin{cases} x' = ax - \varepsilon by \\ y' = bx + \varepsilon ay \end{cases}, \tag{4.9}$$

where $a^2 + b^2 = 1$ and $\varepsilon = \pm 1$.

Conversely, any system of equation of the form (4.9) represents an isometry of $\mathcal{E}_2$, having O as fixed point.

Proof: Take the point $A(1,0)$ and let $A' = f(A)$, with $A'(a,b)$. Let $M(x,y)$ be an arbitrary point and $M' = f(M)$, $M'(x',y')$. Since $1 = |OA| = |OA'|$, then $a^2 + b^2 = 1$.
On the other hand,

$$\begin{cases} |OM| = |OM'| \\ |AM| = |A'M'| \end{cases} \Longleftrightarrow \begin{cases} x^2 + y^2 = x'^2 + y'^2 \\ (x-1)^2 + y^2 = (x'-a)^2 + (y'-b)^2 \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} x = ax' + by' \\ x^2 + y^2 = x'^2 + y'^2 \end{cases}$$

Replacing x in the second equation, one gets

$$(ax' + by')^2 + y^2 = x'^2 + y'^2,$$

or

$$y^2 = (1-a^2)x'^2 - 2abx'y' + (1-b^2)y'^2 = b^2x'^2 - 2abx'y' + a^2y'^2 = (bx' - ay')^2.$$

One obtains the system of equations

$$\begin{cases} x = ax' + by' \\ \varepsilon y = -bx' + ay' \end{cases}, \quad \varepsilon = \pm 1,$$

which gives (4.9).

Conversely, it is clear that $O(0,0)$ is a fixed point, since, for $x = y = 0$, one obtains $x' = y' = 0$.

Let $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ be two arbitrary points and $M_1'(x_1', y_1')$, $M_2'(x_2', y_2')$ be their images, whose coordinates are given through (4.9). Then

$$|M_1'M_2'|^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2 =$$

$$= [a(x_2 - x_1) - \varepsilon b(y_2 - y_1)]^2 + [b(x_2 - x_1) + \varepsilon a(y_2 - y_1)]^2 =$$

$$= (a^2 + b^2)[(x_2 - x_1)^2 + (y_2 - y_1)^2] = |M_1M_2|^2,$$

therefore $|M_1'M_2'| = |M_1M_2|$ and the equations (4.9) define an isometry. □

Let $f \in \text{Iso}(\mathcal{E}_2)$ be an arbitrary isometry of $\mathcal{E}_2$. Suppose that $f(O) = O'$. The product $g = t^{-1}_{\overline{OO'}} \circ g$ is an isometry and $g(O) = O$, therefore O is a fixed point for g. Using Propositions 4.5.3 and 4.5.4, one obtains:

Theorem 4.5.5. *The equations of an arbitrary isometry of $\mathcal{E}_2$ are*

$$\begin{cases} x' = ax - \varepsilon by + x_0 \\ y' = bx + \varepsilon ay + y_0 \end{cases}, \tag{4.10}$$

where $a^2 + b^2 = 1$ and $\varepsilon = \pm 1$.

Remark: The equations (4.10) can be written in the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -\varepsilon b \\ b & \varepsilon a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{4.11}$$

where $a^2 + b^2 = 1$ and $\varepsilon = \pm 1$.

The isometries can be characterized through their fixed points. A point $M(x, y)$ is fixed if and only if its coordinates satisfy the system

$$\begin{cases} (a - 1)x - \varepsilon by = -x_0 \\ bx + (\varepsilon a - 1)y = -y_0 \end{cases}. \tag{4.12}$$

The discriminant of (4.12) is $\Delta = \begin{vmatrix} a - 1 & -\varepsilon b \\ b & \varepsilon a - 1 \end{vmatrix} = (1 - a)(\varepsilon - 1)$.

- If $\varepsilon = -1$, then (4.12) has an infinity of solutions, and f is the product between an axial symmetry and a translation;

- If $\varepsilon = 1$, $a \neq 1$, then (4.12) has a unique solution, and f is the product between a rotation of angle $\theta = \arccos a$, centered at the fixed point of f, and a translation. The equations of f become

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}; \tag{4.13}$$

- If $\varepsilon = 1$, $a = 1$, then $b = 0$ and f is a translation.

4.5.1 Exercises

1) Let $A(-1,2)$ and $B(2,-1)$ be two points and $d: 2x+3y-6=0$ a line on $\mathcal{E}_2$, let $\overline{v}(2,3)$ be a vector, $C = t_{\overline{v}}(A)$, $D = t_{\overline{v}}(B)$ and $b = t_{\overline{v}}(a)$.

   a) Find the coordinates of C, D and the equation of the line b;

   b) Find the coordinates of the intersection between CD and b.

2) Let $A(2,1)$ and $B(6,4)$ be two points and $a: x=3$, $b: x=4$ be two lines on $\mathcal{E}_2$.

   a) Find two points $M \in a$ and $N \in b$, such that $|AM| = |BN|$ and the line MN is parallel to the first bisector $x-y=0$;

   b) Find the points $P \in a$, $Q \in b$, such that $|AP| + |PQ| + |QB|$ is minimal.

3) Let a be a fixed line and $A \in a$ be fixed. A variable circle of given radius r passes through A and intersects the line a at M. Let P be a point on the circle, such that $[AP] \equiv [MP]$.

   a) Find the geometric locus of P;

   b) Find the geometric locus of the orthocenter of the triangle ΔAMP.

4) Find the symmetrical of the point $P(4,2)$ with respect to the line $d: x+2y-4=0$.

5) Find the coordinates of the symmetrical of $M(3,-2)$ with respect to $P(2,1)$.

6) Find the coordinates of a point $M \in Ox$, such that the sum of its distances to $A(1,2)$ and $B(3,4)$ is minimum.

7) The point $A(-4,5)$ is the vertex of a square having a diagonal contained into the line $d: 7x-y+8=0$. Find the coordinates of the vertices of the square.

8) Find the equations of the symmetricals of the line $d: 6x+5y-15=0$ with respect to Ox, Oy and O.

9) Find the equation of the symmetrical of the line $d: -x+2y-1=0$ with respect to $a: x-y=0$.

Chapter 5

Conics

5.1 The Circle

5.1.1 Definition

A *circle* is a closed plane curve, defined as the geometric locus of the points at a given distance R from a point I. The point I is the *center* of the circle and the number R is the *radius* of the circle. We shall denote the circle of center I and radius R by $\mathcal{C}(I, R)$.

In order to determine the equation of the circle, suppose that xOy is an associated Cartesian system of coordinates in $\mathcal{E}_2$, and $I(a, b)$. An arbitrary point $M(x, y)$ belongs to $\mathcal{C}(I, R)$ if and only if $|MI| = R$.

Figure 5.1:

Hence, $\sqrt{(x-a)^2 + (y-b)^2} = R$, or

$$(x-a)^2 + (y-b)^2 = R^2. \tag{5.1}$$

The equation (5.1) represents the equation of the circle centered at $I(a, b)$ and of radius R.

Remark: In a Cartesian system of coordinates, the equation

$$x^2 + y^2 - 2ax - 2by + c = 0 \tag{5.2}$$

represents either a circle, or a point, or the empty set. Indeed, (5.2) can be written in the equivalent form

$$(x - a)^2 + (y - b)^2 = a^2 + b^2 - c.$$

- If $a^2 + b^2 - c > 0$, then (5.2) is the equation of the circle of center $I(a, b)$ and radius $R = \sqrt{a^2 + b^2 - c}$;
- If $a^2 + b^2 - c = 0$, then (5.2) represents the point $I(a, b)$ (the circle is degenerated to its center);
- If $a^2 + b^2 - c < 0$, then (5.2) is the empty set (or an imaginary circle).

The equation (5.2), $x^2 + y^2 - 2ax - 2by + c = 0$, with $a^2 + b^2 - c > 0$, is said to be the *general* equation of the circle.

5.1.2 The Circle Determined by Three Points

Given three noncollinear points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ and $M_3(x_3, y_3)$, there exists a unique circle passing through them. Suppose that the circle determined by $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ and $M_3(x_3, y_3)$ has the general equation

$$x^2 + y^2 - 2ax - 2by + c = 0,$$

with $a^2 + b^2 - c = 0$. Since the three points are on the circle, one obtains the system of equations (with variables a, b and c)

$$\begin{cases} x^2 + y^2 - 2ax - 2by + c = 0 \\ x_1^2 + y_1^2 - 2ax_1 - 2by_1 + c = 0 \\ x_2^2 + y_2^2 - 2ax_2 - 2by_2 + c = 0 \\ x_3^2 + y_3^2 - 2ax_3 - 2by_3 + c = 0 \end{cases},$$

which has to be compatible, so that

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{5.3}$$

The equation (5.3) is the equation of the circle determined by three points.

It follows immediately that four points $M_i(x_i, y_i)$, $i = \overline{1,4}$, belong to a circle if and only if

$$\begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0. \tag{5.4}$$

5.1.3 Intersection of a Circle and a Line

Let $\mathcal{C}$ be a circle and d be a line on $\mathcal{E}_2$. One may choose a system of coordinates having the center at the center of the circle, so that the equation of $\mathcal{C}$ is $x^2 + y^2 - R^2 = 0$. Let $d : y = mx + n$.

The intersection between $\mathcal{C}$ and d is given by the solutions of the system of equations

$$\begin{cases} x^2 + y^2 - R^2 = 0 \\ y = mx + n \end{cases}.$$

By substituting y in the equation of the circle, one obtains

$$(1 + m^2)x^2 + 2mnx + n^2 - R^2 = 0.$$

The discriminant of this second degree equation is

$$\Delta = 4(R^2 + m^2R^2 - n^2).$$

- If $R^2 + m^2R^2 - n^2 < 0$, then there are no intersection points between $\mathcal{C}$ and d. The line is *exterior* to the circle;

- If $R^2 + m^2R^2 - n^2 = 0$, then there is a double point (a *tangency* point) between $\mathcal{C}$ and d. The line is *tangent* to the circle. The coordinates of the tangency point are $\left(-\dfrac{mn}{1+m^2}, \dfrac{n}{1+m^2}\right)$;

- If $R^2 + m^2R^2 - n^2 > 0$, then there are two intersection points between $\mathcal{C}$ and d. The line is *secant* to the circle. If x_1 and x_2 are the roots of the above equation, then the intersection points between $\mathcal{C}$ and d are $P_1(x_1, mx_1 + n)$ and $P_2(x_2, mx_2 + n)$.

5.1.4 The Tangent to a Circle

The Tangent Having a Given Direction

Let $\mathcal{C}$ be the circle of equation $x^2 + y^2 - R^2 = 0$ and $m \in \mathbb{R}$ a given real number. There are two lines, having the angular coefficient m, and which are tangent to $\mathcal{C}$. We saw, in the previous paragraph, that a line $d : y = mx + n$ is tangent to $\mathcal{C}$ if and only if $R^2 + m^2R^2 - n^2 - 0$. Then, the equations of the two tangent lines of direction m are

$$y = mx \pm R\sqrt{1 + m^2}. \tag{5.5}$$

The Tangent to a Circle at a Point of the Circle

Let $\mathcal{C} : x^2 + y^2 - r^2 = 0$ be a circle and $P_0(x_0, y_0)$ be a point on $\mathcal{C}$. The tangent at P_0 to $\mathcal{C}$ is a line from the bundle of lines $y - y_0 = m(x - x_0)$, $m \in \mathbb{R}$, having the vertex P. On the other hand, the tangent has to be of the form (5.5): $y = mx \pm R\sqrt{1 + m^2}$. Then, the angular coefficient m must verify

$$\begin{cases} y - y_0 = m(x - x_0) \\ y = mx \pm R\sqrt{1 + m^2} \end{cases},$$

hence

$$(y_0 - mx_0)^2 = R^2(1 + m^2).$$

But $x_0^2 + y_0^2 = R^2$ (since $P_0 \in \mathcal{C}$) and one obtains $(mx_0 - y_0)^2 = 0$. Therefore $m = -\dfrac{x_0}{y_0}$ (one may suppose that $y_0 \neq 0$; otherwise, one gets the tangent at the point $(R, 0)$, which is of equation $x = R$). Replacing m in the equation of the bundle, one obtains

$$y - y_0 = -\frac{x_0}{y_0},$$

or

$$x_0x + y_0y - (x_0^2 + y_0^2) = 0.$$

Again, $x_0^2 + y_0^2 = R^2$, and the equation of the tangent line to $\mathcal{C}$ at the point $P_0 \in \mathcal{C}$ is

$$x_0x + y_0y - R^2 = 0. \tag{5.6}$$

Remark: The equation of the line OP_0 is $y = \dfrac{y_0}{x_0}x$. Then, the product of the angular coefficients of OP_0 and of the tangent at P_0 is -1, meaning that the tangent at a point to a circle is orthogonal on the radius which corresponds to the point.

5.1.5 Intersection of Two Circles

Given two circles,

$$\mathcal{C}_1 : x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0$$

and

$$\mathcal{C}_2 : x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0,$$

the system of equations

$$\begin{cases} x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0 \\ x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0 \end{cases}$$

gives informations about the intersection of the two circles. The previous system is equivalent to

$$\begin{cases} x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0 \\ 2(a_2 - a_1)x + 2(b_2 - b_1)y - (c_2 - c_1) = 0 \end{cases}$$

which will give rise to a second degree equation, having the discriminant Δ.

- If $\Delta > 0$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are *secant* (they have two intersection points);
- If $\Delta = 0$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are *tangent* (they have one *tangency* point);
- If $\Delta < 0$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ have no intersection points.

5.1.6 Exercises

1) Find the equation of the circle of diameter $[AB]$, where $A(1, 2)$ and $B(-3, -1)$.

2) Find the equation of the circle of center $I(2, -3)$ and radius $R = 7$.

3) Find the equation of the circle of center $I(-1, 2)$ and which passes through $A(2, 6)$.

4) Find the equation of the circle centered at the origin and tangent to $d : 3x - 4y + 20 = 0$.

5) Find the equation of the circle passing through $A(3, 1)$ and $B(-1, 3)$ and having the center on the line $d : 3x - y - 2 = 0$.

6) Find the equation of the circle determined by $A(1, 1)$, $B(1, -1)$ and $C(2, 0)$.

7) Find the equation of the circle tangent to both $d_1 : 2x + y - 5 = 0$ and $d_2 : 2x + y + 15 = 0$, if the tangency point with d_1 is $M(3, 1)$.

8) Determine the position of the point $A(1, -2)$ relative to the circle $\mathcal{C} : x^2 + y^2 - 8x - 4y - 5 = 0$.

9) Find the intersection between the line $d : 7x - y + 12 = 0$ and the circle $\mathcal{C} : (x - 2)^2 + (y - 1)^2 - 25 = 0$.

10) Determine the position of the line $d : 2x - y - 3 = 0$ relative to the circle $\mathcal{C} : x^2 + y^2 - 3x + 2y - 3 = 0$.

11) Find the equation of the tangent to $\mathcal{C} : x^2 + y^2 - 5 = 0$ at the point $A(-1, 2)$.

12) Find the equations of the tangent lines to $\mathcal{C} : x^2 + y^2 + 10x - 2y + 6 = 0$, parallel to $d : 2x + y - 7 = 0$.

13) Find the equations of the tangent lines to $\mathcal{C} : x^2 + y^2 - 2x + 4y = 0$, orthogonal on $d : x - 2y + 9 = 0$.

14) Let $\mathcal{C}_\lambda : x^2 + y^2 + \lambda x + (2\lambda + 3)y = 0$, $\lambda \in \mathbb{R}$, be a family of circles. Prove that the circles from the family have two fixed points.

15) Find the geometric locus of the points in the plane for which the sum of the squares of the distances to the sides of an equilateral triangle is constant.

16) Let P and Q be two fixed points and d a line, orthogonal on PQ. Two variable orthogonal lines, passing through P, cut d at A, respectively B. Find the geometric locus of the orthogonal projection of the point A on the line BQ.

17) Two circles of centers O, respectively O', intersect each other at A and B. A variable line passing through A cuts the two circles at C, respectively C'. Find the geometric locus of the intersection point between the lines OC and $O'C'$.

5.2 The Ellipse

5.2.1 Definition

An *ellipse* is a plane curve, defined to be the geometric locus of the points in the plane, whose distances to two fixed points have a constant sum.

The two fixed points are called the *foci* of the ellipse and the distance between the foci is the *focal distance*.

Let F and F' be the two foci of an ellipse and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition of the ellipse is $2a$. If M is an arbitrary point of the ellipse, it must verify the condition

$$|MF| + |MF'| = 2a.$$

One may chose a Cartesian system of coordinates centered at the midpoint of the segment $[F'F]$, so that $F(c, 0)$ and $F'(-c, 0)$.

Figure 5.2:

Remark: In $\Delta MFF'$ holds $|MF| + |MF'| > |FF'|$, hence $2a > 2c$. Then, the constants a and c must verify $a > c$.

Let us determine the equation of an ellipse. Starting with the definition, $|MF| + |MF'| = 2a$, or

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

This is equivalent to

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

and

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2.$$

One obtains

$$a\sqrt{(x+c)^2 + y^2} = cx + a^2,$$

which gives

$$a^2(x^2 + 2xc + c^2) + a^2y^2 = c^2x^2 + 2a^2cx + a^2,$$

or

$$(a^2 - c^2)x^2 + a^2y^2 - a^2(a^2 - c^2) = 0.$$

Denoting $a^2 - c^2 = b^2$ (possible, since $a > c$), one has

$$b^2x^2 + a^2y^2 - a^2b^2 = 0.$$

Dividing by a^2b^2, one obtains the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0. \tag{5.7}$$

Remark: The equation (5.7) is equivalent with

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}; \qquad x = \pm\frac{a}{b}\sqrt{b^2 - y^2},$$

which means that the ellipse is symmetrical with respect to both Ox and Oy. In fact, the line FF', determined by the foci of the ellipse, and the perpendicular line on the midpoint of the segment $[FF']$ are axes of symmetry for the ellipse. Their intersection point, which is the midpoint of $[FF']$, is the center of symmetry of the ellipse, or, simply, its *center*.

In order to sketch the graph of the ellipse, remark that is it enough to represent the function

$$f : [-a, a] \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{a^2 - x^2},$$

and to complete the ellipse by symmetry with respect to Ox.

One has

$$f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2 - x^2}}, \qquad f''(x) = -\frac{ab}{(a^2 - x^2)\sqrt{a^2 - x^2}}.$$

x	$-a$		0		a
$f'(x)$	$\vert$	$+\ +\ +$	0	$-\ -\ -$	$\vert$
$f(x)$	0	$\nearrow$	b	$\searrow$	0
$f''(x)$	$\vert$	$-\ -\ -$	$-$	$-\ -\ -$	$\vert$

The graph of the ellipse is presented in Figure 5.3.

Figure 5.3:

Remarks:

- In particular, if $a = b$ in (5.7), one obtains the equation $x^2 + y^2 - a^2 = 0$ of the circle centered at the origin and of radius a. This happens when $c = 0$, i.e. when the foci coincide, so that the circle may be seen as an ellipse whose foci are identical.

- All the considerations can be done in a similar way, by taking the foci of the ellipse on Oy. One obtains a similar equation for such an ellipse.

The number $e = \dfrac{c}{a}$ is called the *eccentricity* of the ellipse. Since $a > c$, then $0 < e < 1$, hence any ellipse has the eccentricity smaller that 1. On the other hand, $e^2 = \dfrac{c^2}{a^2} = 1 - \left(\dfrac{b}{a}\right)^2$, so that e gives informations about the shape of the ellipse. When e is closer and closer to 0, then the ellipse is "closer and closer" to a circle; and when e is closer to 1, then the ellipse is flattened to Ox.

5.2.2 Intersection of a Line and an Ellipse

Given an ellipse $\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ and a line $d : y = mx + n$, their intersection is given by the solutions of the system of equations

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases},$$

or, by replacing y in the equation of the ellipse,

$$(a^2m^2+b^2)x^2+2a^2mnx+a^2(n^2-b^2)=0.$$

The discriminant Δ of the last equation is given by

$$\Delta=4[a^4m^2n^2-a^2(a^2m^2+b^2)(n^2-b^2)].$$

- If $\Delta<0$, then d does not intersect $\mathcal{E}$. The line is *exterior* to the ellipse;
- If $\Delta=0$, then the line is *tangent* to the ellipse. There is a *tangency* point between d and $\mathcal{E}$;
- If $\Delta>0$, then there are two intersection points between d and $\mathcal{E}$. The line is *secant* to the ellipse.

5.2.3 The Tangent to an Ellipse

The Tangent of a Given Direction

If $\mathcal{E}:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ is an ellipse and $m\in\mathbb{R}$ a given real number, there exist exactly two lines, having the angular coefficient m and tangent to $\mathcal{E}$. Since a line $d:y=mx+n$ is tangent to the ellipse if and only if $a^4m^2n^2-a^2(a^2m^2+b^2)(n^2-b^2)=0$, then $n=\pm\sqrt{a^2m^2+b^2}$. The equations of the tangent lines of direction m are

$$y=mx\pm\sqrt{a^2m^2+b^2}. \tag{5.8}$$

The Tangent to an Ellipse at a Point of the Ellipse

Let $\mathcal{E}:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ be an ellipse and $P_0(x_0,y_0)$ be a point of $\mathcal{E}$. Suppose that $y_0>0$, so that P_0 is situated on the graph of the function $f:[-a,a]\to\mathbb{R}$, $f(x)=\dfrac{b}{a}\sqrt{a^2-b^2}$. The angular coefficient of the tangent at P_0 to $\mathcal{E}$ is

$$f'(x_0)=-\frac{b}{a}\frac{x_0}{\sqrt{a^2-b^2}}=-\frac{b^2x_0}{a^2y_0}.$$

If $y_0<0$, a similar argument shows that the angular coefficient of the tangent at P_0 is still $-\dfrac{b^2x_0}{a^2y_0}$.

The equation of the tangent to $\mathcal{E}$ at P_0 is

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0),$$

equivalent to

$$b^2 x_0(x - x_0) + a^2 y_0(y - y_0) = 0,$$

or

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right) = 0.$$

Since P_0 belongs to the ellipse, then $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$, and the equation of the tangent to the ellipse at the point P_0 is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - 1 = 0. \tag{5.9}$$

5.2.4 Exercises

1) Determine the coordinates of the foci of the ellipse $9x^2 + 25y^2 - 225 = 0$.

2) Sketch the graph of $y = -\dfrac{3}{4}\sqrt{16 - x^2}$.

3) Find the intersection points between the line $x + 2y - 7 = 0$ and the ellipse $x^2 + 3y^2 - 25 = 0$.

4) Find the position of the line $2x + y - 10 = 0$ relative to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} - 1 = 0$.

5) Find the equation of the tangent to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$, orthogonal on the line $d : 2x - 2y - 13 = 0$.

6) Find the equations of the tangent lines at $\dfrac{x^2}{25} + \dfrac{y^2}{16} - 1 = 0$, passing through $P_0(10, -8)$.

7) Find the geometric locus of the orthogonal projections of a focus of an ellipse on the tangent lines to the ellipse.

8) Let d_1 and d_2 be two variable orthogonal lines, passing through the point $A(a, 0)$ of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and let P_1 and P_2 be the intersection points of these two lines and the ellipse. Prove that the line P_1P_2 passes through a fixed point.

5.3 The Hyperbola

5.3.1 Definition

A *hyperbola* is a plane curve, defined as the geometric locus of the points in the plane, whose distances to two fixed points have a constant difference.

The two fixed points are called the *foci* of the hyperbola, and the distance between the foci is the *focal distance*.

Denote by F and F' the foci of the hyperbola and let $|FF'| = 2c$ be the focal distance. Suppose that the constant in the definition is $2a$. If $M(x, y)$ is an arbitrary point of the hyperbola, then

$$||MF| - |MF'|| = 2a.$$

Choose a Cartesian system of coordinates, having the center at the midpoint of the segment $[FF']$ and such that $F(c, 0)$, $F'(-c, 0)$.

Figure 5.4:

Remark: In the triangle $\Delta MFF'$, $||MF| - |MF'|| < |FF'|$, so that $a < c$.

The metric relation $|MF| - |MF'| = \pm 2a$ becomes

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a,$$

or,

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}.$$

This is

$$x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \Longleftrightarrow$$

$$\Longleftrightarrow cx + a^2 = \pm a\sqrt{(x+c)^2 + y^2} \Longleftrightarrow$$

$$\Longleftrightarrow c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 \Longleftrightarrow$$

$$\Longleftrightarrow (c^2-a^2)x^2 - a^2y^2 - a^2(c^2-a^2) = 0.$$

Denote $c^2 - a^2 = b^2$ (possible, since $c > a$) and one obtains the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0. \tag{5.10}$$

Remark: The equation (5.10) is equivalent to

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}; \qquad x = \pm\frac{a}{b}\sqrt{y^2 + b^2}.$$

Then, the coordinate axes are axes of symmetry for the hyperbola. Their intersection point is the *center* of the hyperbola.

To sketch the graph of the hyperbola, is it enough to represent the function

$$f : (-\infty, -a] \cup [a, \infty) \to \mathbb{R}, \qquad f(x) = \frac{b}{a}\sqrt{x^2 - a^2},$$

by taking into account that the hyperbola is symmetrical with respect to Ox.

Since $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \frac{b}{a}$ and $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = -\frac{b}{a}$, it follows that $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are asymptotes of f.

One has, also,

$$f'(x) = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}, \qquad f''(x) = -\frac{ab}{(x^2 - a^2)\sqrt{x^2 - a^2}}.$$

x	$-\infty$		$-a$		a		∞
$f'(x)$	$-$	$- - -$	$\mid$	$/ / /$	$\mid$	$+ + +$	$+$
$f(x)$	∞	$\searrow$	$0\mid$	$/ / /$	$\mid 0$	$\nearrow$	∞
$f''(x)$	$-$	$- - -$	$\mid$	$/ / /$	$\mid$	$- - -$	$-$

The graph of the hyperbola is presented in Figure 5.5.

Remarks:

- If $a = b$, the equation of the hyperbola becomes $x^2 - y^2 = a^2$. In this case, the asymptotes are the bisectors of the system of coordinates and one deals with an *equilateral* hyperbola.
- As in the case of an ellipse, one can consider the hyperbola having the foci on Oy.

Figure 5.5:

The number $e = \dfrac{c}{a}$ is called the *eccentricity* of the hyperbola. Since $c > a$, then the eccentricity is always greater than 1. Moreover,

$$e^2 = \frac{c^2}{a^2} = 1 + \left(\frac{b}{a}\right)^2,$$

hence e gives informations about the shape of the hyperbola. For e closer to 1, the hyperbola has the branches closer to Ox.

5.3.2 Intersection of a Hyperbola and a Line

Let $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ be a hyperbola and $d : y = mx + n$ be a line in $\mathcal{E}_2$. Their intersection is given by the system of equations

$$\begin{cases} \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0 \\ y = mx + n \end{cases}.$$

By substituting y in the first equation, one obtains

$$(a^2m^2 - b^2)x^2 + 2a^2mnx + a^2(n^2 + b^2) = 0. \tag{5.11}$$

- If $a^2m^2 - b^2 = 0$, (or $m = \pm\dfrac{b}{a}$), then the equation (5.11) becomes

$$\pm 2bnx + a(n^2 + b^2) = 0.$$

- If $n = 0$, there are no solutions (this means, geometrically, that the two asymptotes do not intersect the hyperbola);
- If $n \neq 0$, there exists a unique solution (geometrically, a line d, which is parallel to one of the asymptotes, intersects the hyperbola at exactly one point);

- If $a^2m^2 = b^2 \neq 0$, then the discriminant of the equation (5.11) is

$$\Delta = 4[a^4m^2n^2 - a^2(a^2m^2 - b^2)(n^2 + b^2)].$$

  - If $\Delta < 0$, then the line does not intersect the hyperbola;
  - If $\Delta = 0$, then the line is *tangent* to the hyperbola (they have a double intersection point);
  - If $\Delta > 0$, then the line and the hyperbola have two intersection points.

5.3.3 The Tangent to a Hyperbola

The Tangent of a Given Direction

The line $d : y = mx + n$ is tangent to the hyperbola $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ if the discriminant Δ of the equation (5.11) is zero, which is equivalent to $a^2m^2 - n^2 - b^2 = 0$.

- If $a^2m^2 - b^2 \geq 0$, i.e. $m \in \left(-\infty, -\dfrac{b}{a}\right] \cup \left[\dfrac{b}{a}, \infty\right)$, then $n = \pm\sqrt{a^2m^2 - b^2}$.
  The equations of the tangent lines to $\mathcal{H}$, having the angular coefficient m are

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{5.12}$$

- If $a^2m^2 - b^2 < 0$, there are no tangent lines to $\mathcal{H}$, of angular coefficient m.

The Tangent at a Point of the Hyperbola

One can prove, as in the case of the ellipse that, if $\mathcal{H} : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ is a hyperbola, and $P_0(x_0, y_0)$ is a point of $\mathcal{H}$, then the equation of the tangent to $\mathcal{H}$ at P_0 is

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} - 1 = 0. \tag{5.13}$$

5.3.4 Exercises

1) Find the foci of the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} - 1 = 0$.

2) Find the area of the triangle determined by the asymptotes of the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{9} - 1 = 0$ and the line $d : 9x + 2y - 24 = 0$.

3) Find the intersection points between the hyperbola $\dfrac{x^2}{20} - \dfrac{y^2}{5} - 1 = 0$ and the line $d : 2x - y - 10 = 0$.

4) Find the equations of the tangent lines to the hyperbola $\dfrac{x^2}{20} - \dfrac{y^2}{5} - 1 = 0$, which are orthogonal to $d : 4x + 3y - 7 = 0$.

5) Find the equations of the tangent lines to the hyperbola $\dfrac{x^2}{3} - \dfrac{y^2}{5} - 1 = 0$, passing through $P(1, -5)$.

6) Find the geometric locus of the orthogonal projections of a focus of a hyperbola on the tangent lines to the hyperbola.

5.4 The Parabola

5.4.1 Definition

The *parabola* is a plane curve defined to be the geometric locus of the points in the plane, whose distance to a fixed line d is equal to its distance to a fixed point F.

The line d is the *director line* and the point F is the *focus*. The distance between the focus and the director line is denoted by p and represents the *parameter* of the parabola.

Consider a Cartesian system of coordinates xOy, in which $F\left(\dfrac{p}{2}, 0\right)$ and $d : x = -\dfrac{p}{2}$. If $M(x, y)$ is an arbitrary point of the parabola, then it verifies

$$|MN| = |MF|,$$

where N is the orthogonal projection of M on Oy.

Figure 5.6:

Thus, the coordinates of a point of the parabola verify

$$\sqrt{\left(x+\frac{p}{2}\right)^2+0}=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}\Leftrightarrow$$

$$\Leftrightarrow\left(x+\frac{p}{2}\right)^2=\left(x-\frac{p}{2}\right)^2=y^2\Leftrightarrow$$

$$\Leftrightarrow x^2+px+\frac{p^2}{4}=x^2-px+\frac{p^2}{4}+y^2,$$

and the equation of the parabola is

$$y^2=2px. \tag{5.14}$$

Remark: The equation (5.14) is equivalent to $y=\pm\sqrt{2px}$, so that the parabola is symmetrical with respect to Ox.

Representing the graph of the function $f:[0,\infty)\to[0,\infty)$ and using the symmetry of the curve with respect to Ox, one obtains the graph of the parabola (see Figure (5.7). One has

$$f'(x)=\frac{p}{\sqrt{2px_0}};\qquad f''(x)=-\frac{p}{2x\sqrt{2x}}.$$

x	0		∞
$f'(x)$	$\vert$	$+\ +\ +$	$+$
$f(x)$	0	$\nearrow$	∞
$f''(x)$	$-$	$-\ -\ -$	$-$

Figure 5.7:

5.4.2 Intersection of a Parabola and a Line

Let $\mathcal{P} : y^2 = 2px$ be a parabola, $d : y = mx + n$ $(m \neq 0)$ be a line and

$$\begin{cases} y^2 = 2px \\ y = mx + n \end{cases}$$

be the system determined by their equations. This leads to a second degree equation

$$m^2x^2 + 2(mn - p)x + n^2 = 0,$$

having the discriminant

$$\Delta = 4p(2mn - p) \tag{5.15}$$

- If $\Delta < 0$, then the line does not intersect the parabola;
- If $\Delta > 0$, then there are two intersection points between the line and the parabola;
- If $\Delta = 0$, then the line is *tangent* to the parabola and they have a unique intersection point.

5.4.3 The Tangent to a Parabola

The Tangent of a Given Direction

A line $d : y = mx + n$ (with $m \neq 0$) is tangent to the parabola $\mathcal{P} : y^2 = 2px$ if the discriminant Δ which appears in (5.15) is zero, i.e. $2mn = p$. Then, the equation of the tangent line to $\mathcal{P}$, having the angular coefficient m, is

$$y = mx + \frac{p}{2m}. \tag{5.16}$$

The Tangent to a Parabola at a Point of the Parabola

Let $\mathcal{P} : y^2 = 2px$ be a parabola and $P_0(x_0, y_0)$ be a point of $\mathcal{P}$. Suppose that $y_0 > 0$, so that the point P_0 belongs to the graph of the function $f : [0, \infty) \to [0, \infty)$, $f(x) = \sqrt{2px}$. The angular coefficient of the tangent at P_0 to the curve is

$$f'(x_0) = \frac{p}{\sqrt{2px_0}} = \frac{p}{y_0}.$$

A similar computation leads to the angular coefficient of the tangent for $y_0 < 0$, which is still $\frac{p}{y_0}$.

The equation of the tangent at P_0 to $\mathcal{P}$ is

$$y - y_0 = f'(x_0)(x - x_0),$$

or, replacing $f'(x_0)$,

$$y - y_0 = \frac{p}{y_0}(x - x_0) \Leftrightarrow$$

$$\Leftrightarrow yy_0 - y_0^2 = p(x - x_0) \Leftrightarrow$$

$$yy_0 - 2px_0 = p(x - x_0),$$

hence the equation of the tangent is

$$yy_0 = p(x + x_0). \tag{5.17}$$

5.4.4 Exercises

1) Find the focus and the director line of the parabola $y^2 - 24x = 0$.

2) Find the equation of the parabola having the focus $F(-7, 0)$ and the director line $x - 7 = 0$.

3) Find the equation of the tangent line to the parabola $y^2 - 8x = 0$, parallel to $d : 2x + 2y - 3 = 0$.

4) Find the equations of the tangent lines to the parabola $y^2 - 36x = 0$, passing through $P(2, 9)$.

5) Find the equation of the tangent line to $y^2 - 4x = 0$ at the point $P(1, 2)$.

6) Let $\mathcal{P}_1 : y^2 - 2px = 0$ and $\mathcal{P}_2 : y^2 - 2qx = 0$ be two parabolas, with $0 < q < p$. A mobile tangent to $\mathcal{P}_2$ intersects $\mathcal{P}_1$ at M_1 and M_2. Find the geometric locus of the midpoint of the segment $[M_1 M_2]$.

7) Find the geometric locus of the orthogonal projections of the focus of a parabola on the tangent lines to the parabola.

8) Let A, B and C be three distinct points on the parabola of equation $y^2 = 2px$. The tangent lines at A, B respectively C to the parabola determine a triangle $A'B'C'$. Prove that the line passing through the centers of gravity of the triangles ΔABC and $\Delta A'B'C'$ is parallel to Ox.

9) Let a, b and c be the tangent lines at three distinct point of a parabola and ABC the triangle determined by the tangents. Prove that the focus of the parabola belongs to the circumscribed circle of the triangle ΔABC.

5.5 Conics Defined Through a General Equation

5.5.1 Definition

Let be given the Euclidean plane $\mathcal{E}_2$ and a Cartesian system of coordinates xOy, associated to it. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function, defined through

$$f(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00}, \tag{5.18}$$

with $a_{11}^2 + a_{12}^2 + a_{22}^2 \neq 0$.

The set $\Gamma = \{P(x, y) : f(x, y) = 0\}$ is called *algebraic curve* of degree 2, or *curve* in $\mathcal{E}_2$.

Consider the real numbers

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{10} \\ a_{12} & a_{22} & a_{20} \\ a_{10} & a_{20} & a_{00} \end{vmatrix}; \qquad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}; \qquad I = a_{11} + a_{22}. \tag{5.19}$$

We shall prove that, after making some changes of coordinates (given, in fact, by a rotation and, eventually, a translation of the original system), the equation (5.18) turns into an equation which can be identified to be of one of the conics already studied.

Remark: If, in the original system of coordinates, a point P is of coordinates $P(x, y)$, then, after a rotation and a translation, the coordinates of its image are

$$\begin{cases} x' = ax - \varepsilon by + x_0 \\ y' = bx + \varepsilon ay + y_0 \end{cases}, \tag{5.20}$$

where $a^2 + b^2 = 1$ and $\varepsilon = \pm 1$.

Suppose that the point P belongs to the conic Γ. Expressing x and y in the system of equation (5.20) and replacing them in (5.18), one obtains some polynomial of degree 2, in variables x' and y'. It can be verified that the numbers Δ', δ' and I', associated to this polynomial, coincide with Δ, δ and, respectively, I.

The numbers Δ, δ and I are the *metric invariants* of the conic Γ.

Theorem 5.5.1.1. *The point $M_0(x_0, y_0)$ is the center of symmetry of the conic Γ if and only if (x_0, y_0) is a critical point of the function f.*

Proof: Let Γ be given by the zeros of $f : \mathbb{R}^2 \to \mathbb{R}$, as in (5.18). After a translation

$$\begin{cases} x = x_0 + x' \\ y = y_0 + y' \end{cases},$$

the equation $f(x, y) = 0$ becomes

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + f'_x(x_0, y_0)x' + f'_y(x_0, y_0)y' + f(x_0, y_0) = 0. \tag{5.21}$$

Suppose that the point M_0 is the center of symmetry of the conic Γ. Then, the origin $(0, 0)$ is the center of symmetry in the new system. Hence, both $P(x', y')$ and $s_O(P)(-x', -y')$ belong to the conic, when $P \in \Gamma$. Thus,

$$\begin{cases} a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + f'_x(x_0, y_0)x' + f'_y(x_0, y_0)y' + f(x_0, y_0) = 0 \\ a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 - f'_x(x_0, y_0)x' - f'_y(x_0, y_0)y' + f(x_0, y_0) = 0 \end{cases}$$

and $f'_x(x_0, y_0)x' + f'_y(x_0, y_0)y' = 0$, for any $P(x', y') \in \Gamma$. Then,

$$\begin{cases} f'_x(x_0, y_0) = 0 \\ f'_y(x_0, y_0) = 0 \end{cases},$$

and $(x_0, y_0) \in C(f)$.

Conversely, if $(x_0, y_0) \in C(f)$, then $f'_x(x_0, y_0) = 0$ and $f'_y(x_0, y_0) = 0$, the equation (5.21) becomes $a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + f(x_0, y_0) = 0$ and, obviously, $O(0, 0)$ is center of symmetry, i.e. $M_0(x_0, y_0)$ is the center of symmetry for Γ. $\square$

5.5.2 Some Classification Algorithms

The centers of symmetry of a conic can give an algorithm to classify the conics. The critical points of f are given by the solutions of the system of equations

$$\begin{cases} f'_x(x_0, y_0) = 0 \\ f'_y(x_0, y_0) = 0 \end{cases} \Longleftrightarrow \begin{cases} a_{11}x_0 + a_{12}y_0 + a_{10} = 0 \\ a_{12}x_0 + a_{22}y_0 + a_{20} \end{cases},$$

with $\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}$. Let $r =$rank $\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ and $r' =$ rank $\begin{pmatrix} a_{11} & a_{12} & a_{10} \\ a_{12} & a_{22} & a_{20} \end{pmatrix}$.

- If $\delta \neq 0$, then $r = r' = 2$ and Γ has a unique center of symmetry. The conics with this property are: the circle, the ellipse, the hyperbola, a pair of concurrent lines, a point, the empty set. The equation of Γ becomes

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + \frac{\Delta}{\delta} = 0. \tag{5.22}$$

- If $\delta = 0$ and $\Delta \neq 0$, then $r = 1$, $r' = 2$ and Γ has no center of symmetry. The parabola has this property.
- If $\delta = 0$ and $\Delta = 0$, then $r = r' = 1$ and Γ has an entire line of centers of symmetry. The conics with this property are: two parallel lines, two identical lines, the empty set.

The conics can, also, be classified through the sign of the invariant δ.

- If $\delta > 0$, the conic is said to be of *elliptical genus*;
- If $\delta < 0$, then the conic is of *hyperbolical genus*;
- If $\delta = 0$, then the conic is of *parabolical genus.*

Remark: A particular case of conics is obtained if $\delta \neq 0$ and, in (5.22), $a_{12} = 0$ and $a_{11} = a_{22} = a \neq 0$. Then, the equation (5.22) becomes

$$ax'^2 + ay'^2 + \frac{\Delta}{\delta} = 0 \Longleftrightarrow x'^2 + y'^2 = -\frac{\Delta}{a\delta}.$$

- If $-\dfrac{\Delta}{a\delta} < 0$, then $\Gamma = \emptyset$;
- If $-\dfrac{\Delta}{a\delta} = 0$, then $\Gamma = \{M_0(x_0, y_0)\}$; the conic is degenerated to one point: its center of symmetry;
- If $-\dfrac{\Delta}{a\delta} > 0$, then Γ is the circle centered at M_0 and of radius $\sqrt{-\dfrac{\Delta}{a\delta}}$.

5.5.3 Methods of Graphical Representation

Let Γ be the conic defined through (5.18) and Δ, δ and I be the invariants of Γ, given by (5.19). We saw that, after a translation, the equation of Γ has been reduced to

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + \frac{\Delta}{\delta} = 0.$$

- If $a_{12} = 0$, then one only makes the translation, and Γ has the form

$$a_{11}x'^2 + a_{22}y'^2 + \frac{\Delta}{\delta} = 0.$$

- If $a_{12} \neq 0$, then, before making the translation, one makes a rotation (which will cancel the a_{12}) and, after, the translation.

For the latter situation, we present here two methods of representation.

The Eigenvalues Method

Take the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, whose determinant is δ. The eigenvalues of this matrix are the solutions of its characteristic equation:

$$\det\,(A - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{12} & a_{22-\lambda} \end{vmatrix} \Leftrightarrow \lambda^2 - I\lambda + \delta = 0. \tag{5.23}$$

The discriminant of the last equation is

$$I^2 - 4\delta = (a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2) = (a_{11}-a_{22})^2 + 4a_{12}^2 > 0, \quad (\text{since } a_{12} \neq 0),$$

so that the equation (5.23) has two real and distinct solutions, λ_1 and λ_2.

Let $\overline{a}_1(u_1, v_1)$ and $\overline{a}_2(u_2, v_2)$ be the eigenvectors associated to the eigenvalues λ_1 respectively λ_2. Their components are, thus, given by the solutions of the systems

$$\begin{cases} (a_{11} - \lambda_1)u_1 + a_{12}v_1 = 0 \\ a_{12}u_1 + (a_{22} - \lambda_1)v_1 = 0 \end{cases}, \quad \text{respectively} \quad \begin{cases} (a_{11} - \lambda_2)u_2 + a_{12}v_2 = 0 \\ a_{12}u_2 + (a_{22} - \lambda_2)v_2 = 0 \end{cases}.$$

Consider the matrix R, whose columns are given by the components of the versors $\overline{e}_1$ and $\overline{e}_2$ of $\overline{a}_1$, respectively $\overline{a}_2$, such that $\det R = 1$ (one might, eventually, replace one of the versors by its opposite, in order to have the determinant equal to 1).

The solution of the matrix equation $\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x' \\ y' \end{pmatrix}$ (which expresses the rotation) reduces the conic to the following form:

$$\lambda_1 x'^2 + \lambda_2 y'^2 + 2a'_{10}x' + 2a'_{20}y' + a'_{00} = 0.$$

This is equivalent with

$$\lambda_1 \left(x' + \frac{a'_{10}}{\lambda_1} \right)^2 + \lambda_2 \left(y' + \frac{a'_{20}}{\lambda_2} \right)^2 + a'_{00} - \frac{a'^2_{10}}{\lambda_1} - \frac{a'^2_{20}}{\lambda_2} = 0.$$

After a translation of equations

$$\begin{cases} x'' = x' + \dfrac{a'_{10}}{\lambda_1} \\ y'' = y' + \dfrac{a'_{20}}{\lambda_2} \end{cases},$$

one obtains the canonical equation of the conic

$$\lambda_1 x'' + \lambda_2 y'' + a = 0,$$

where $a = a'_{00} - \dfrac{a'^2_{10}}{\lambda_1} - \dfrac{a'^2_{20}}{\lambda_2}$.

Example: Let us consider the conic Γ, given by $3x^2 - 4xy - 2x + 4y - 3 = 0$. The associated invariants of Γ are

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -2 & 0 & 2 \\ -1 & 2 & -3 \end{vmatrix} = 8; \qquad \delta = \begin{vmatrix} 3 & -2 \\ -2 & 0 \end{vmatrix} = -4; \qquad I = 3 + 0 = 3.$$

Since $\delta \neq 0$, then Γ has a unique center of symmetry. Its coordinates are given by the solution of the system of equations

$$\begin{cases} 6x - 4y - 2 = 0 \\ -4x + 4 = 0 \end{cases},$$

so that the center of symmetry is $C(1, 1)$.

Since $\delta < 0$, the conic is of hyperbolical genus.

The eigenvalues λ_1 and λ_2 are the solutions of the equation $\lambda^2 - 3\lambda - 4 = 0$, hence $\lambda_1 = -1$ and $\lambda_2 = 4$.

Let us determine the eigenvectors associated to λ_1 and λ_2.

$$\lambda_1 = -1 \qquad \begin{cases} 4u_1 - 2v_1 = 0 \\ -2u_1 + v_1 = 0 \end{cases} \Leftrightarrow \bar{a}_1(\alpha, 2\alpha),\ \alpha \in \mathbb{R}^* \Rightarrow \bar{e}_1\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\lambda_2 = 4 \qquad \begin{cases} -u_2 - 2v_2 = 0 \\ -2u_2 - 4v_2 = 0 \end{cases} \Leftrightarrow \bar{a}_2(-2\beta, \beta),\ \beta \in \mathbb{R}^* \Rightarrow \bar{e}_2\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$$

The matrix of the rotation is given by $R = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$, and $\det R = 1$.

The equations of the rotation are

$$\begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} x' \\ y' \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \Longleftrightarrow \begin{cases} x = \frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' \\ y = \frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y' \end{cases}.$$

Replacing in the equation of the conic, one obtains

$$-x'^2 + 4y'^2 + \frac{6}{\sqrt{5}}x' + \frac{8}{\sqrt{5}}y' - 3 = 0,$$

or

$$-\left(x' - \frac{3}{\sqrt{5}}\right)^2 + 4\left(y' + \frac{1}{\sqrt{5}}y'\right)^2 - 2 = 0.$$

After a translation of equations $\begin{cases} x'' = x' - \frac{3}{\sqrt{5}} \\ y'' = y' + \frac{1}{\sqrt{5}} \end{cases}$, the conic is of the form

$-x''^2 + 4y''^2 - 2 = 0$, so that the canonic equation of the given conic is

$$\frac{x''}{2} - \frac{y''}{\frac{1}{2}} = -1,$$

and it is a hyperbola.

Figure 5.8:

The Transformations Method

One can determine the angle of rotation of the coordinate axes. Let Γ be a conic given by (5.18), with $a_{12} \neq 0$.

Theorem 5.5.3.1. *The angle θ of the rotation r_θ is given by the equation*

$$(a_{11} - a_{22}) \sin 2\theta = 2a_{12} \cos 2\theta. \tag{5.24}$$

Proof: The matrix R is, actually, the matrix of r_θ, so that

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The equations of the rotation are

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x' \\ y' \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \Longleftrightarrow \begin{cases} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{cases}$$

Replacing in the equation of the conic, one obtains

$$f(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = 0 \Leftrightarrow$$

$$\Leftrightarrow a_{11}(x'\cos\theta - y'\sin\theta)^2 + 2a_{12}(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) +$$

$$+ a_{22}(x'\sin\theta + y'\cos\theta)^2 + 2a_{10}(x'\cos\theta - y'\sin\theta) + 2a_{20}(x'\sin\theta + y'\cos\theta) + a_{00} = 0.$$

The coefficient of $x'y'$ in this equation is

$$(a_{11} - a_{22})\sin 2\theta - 2a_{12}\cos 2\theta.$$

But we saw that the effect of the rotation is to cancel this coefficient. Thus

$$(a_{11} - a_{22})\sin 2\theta - 2a_{12}\cos 2\theta = 0. \quad \square$$

Example: Let us take the conic $x^2 + xy + y^2 - 6x - 16 = 0$. The invariants are

$$\Delta = \begin{vmatrix} 1 & \frac{1}{2} & -3 \\ \frac{1}{2} & 1 & 0 \\ -3 & 0 & -16 \end{vmatrix} = -21; \qquad \delta = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4}; \qquad I = 1 + 1 = 2.$$

Since $\delta \neq 0$ and $\delta > 0$, then the conic has a unique center of symmetry and is of elliptic genus. The coordinates of the center of symmetry are given by

$$\begin{cases} 2x + y - 6 = 0 \\ x + 2y = 0 \end{cases},$$

so that the center of symmetry is $C(4, -2)$.

The angle of rotation is given by

$$(1-1)\sin 2\theta = \cos 2\theta \Longleftrightarrow \cos 2\theta = 0 \Longleftrightarrow \theta = \frac{\pi}{4}.$$

The eigenvalues λ_1 and λ_2 are the solutions of the equation $\lambda^2 - 2\lambda + \dfrac{3}{4} = 0$, hence $\lambda_1 = \dfrac{3}{2}$ and $\lambda_2 = \dfrac{1}{2}$.

The equations of the rotation are

$$\begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} x' \\ y' \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \Longleftrightarrow \begin{cases} x = x'\dfrac{\sqrt{2}}{2} - y'\dfrac{\sqrt{2}}{2} \\ y = x'\dfrac{\sqrt{2}}{2} + y'\dfrac{\sqrt{2}}{2} \end{cases}$$

and the conic becomes

$$\frac{3}{2}x'^2 + \frac{1}{2}y'^2 - 3\sqrt{2}x' + 3\sqrt{2}y' - 16 = 0,$$

or

$$\frac{3}{2}(x' - \sqrt{2})^2 + \frac{1}{2}(y' + 3\sqrt{2})^2 - 28 = 0.$$

After a translation of equations $\begin{cases} x'' = x' - \sqrt{2} \\ y'' = y' + 3\sqrt{2} \end{cases}$, one obtains the reduced equation of the conic

$$\frac{x''^2}{\frac{56}{3}} + \frac{y''^2}{56} = 1,$$

the conic being an ellipse.

Figure 5.9:

Conclusions

We can put together all the considerations we have made. Let Γ be the conic given by the zeros of the polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00},$$

with $a_{11}^2 + a_{12}^2 + a_{22}^2 \neq 0$. Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{10} \\ a_{12} & a_{22} & a_{20} \\ a_{10} & a_{20} & a_{00} \end{vmatrix}; \qquad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}; \qquad I = a_{11} + a_{22}.$$

be the invariants of Γ.

<table>
<tr><th colspan="3">Conditions</th><th>The curve</th><th>Transformations</th></tr>
<tr><td rowspan="3">$\Delta = 0$</td><td colspan="2">$\delta > 0$</td><td>$\Gamma = \{(x_0, y_0)\}$</td><td rowspan="7">If $a_{12} = 0$, one makes a lation;

If $a_{12} \neq 0$, one make tation of angle θ, gi $(a_{11}-a_{22}) \sin 2\theta = 2a_{12}$ and, eventually, a trans</td></tr>
<tr><td colspan="2">$\delta = 0$</td><td>two parallel lines, two identical lines, or the empty set</td></tr>
<tr><td colspan="2">$\delta < 0$</td><td>two concurrent lines; if $I = 0$, then the lines are orthogonal.</td></tr>
<tr><td rowspan="4">$\Delta \neq 0$</td><td rowspan="2">$\delta < 0$</td><td>$I\Delta < 0$</td><td>ellipse</td></tr>
<tr><td>$I\Delta > 0$</td><td>$\emptyset$</td></tr>
<tr><td colspan="2">$\delta = 0$</td><td>parabola</td></tr>
<tr><td colspan="2">$\delta < 0$</td><td>hyperbola; if $I = 0$, then the hyperbola is equilateral.</td></tr>
</table>

5.5.4 Exercises

1) Find the canonical equation and sketch the graph of the conic

$$5x^2 + 4xy + 8y^2 - 32x - 56y + 80 = 0.$$

2) Find the canonical equation and sketch the graph of the conic

$$8y^2 + 6xy - 12x - 26y + 11 = 0.$$

3) Find the canonical equation and sketch the graph of the conic

$$x^2 - 4xy + x^2 - 6x + 2y + 1 = 0.$$

4) Discuss the nature of the conics in the family

$$x^2 + \lambda xy + y^2 - 6x - 16 = 0, \quad \lambda \in \mathbb{R}.$$

Chapter 6

Quadric Surfaces

The quadric surfaces can be seen as the three dimensional analogs of the conics. The set S of the points $(x, y, z) \in \mathbb{R}^3$, such that $g(x, y, z) = 0$, where $g \in \mathbb{R}[X, Y, Z]$ is a polynomial, is called *algebraic surface* in $\mathbb{R}^3$. If $\deg(g) = 2$, S is said to be a *quadric surface* in $\mathbb{R}^3$. Hence, the points of a quadric surface are the zeros of the polynomial function

$$g(x,y,z)=a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{13}xz+2a_{23}yz+2a_{10}x+2a_{20}y+2a_{30}x+a_{00}$$

with $a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{12}^2 + a_{23}^2 + a_{13}^2 > 0$.

If $Oxyz$ is a Cartesian system of coordinates, the sections determined by the coordinate planes on a quadric surface are conics, and so are the sections determined by planes which are parallel to the coordinate planes. By studying these sections, one can imagine the shape of a given quadric surface.

The simplest equations for the quadric surfaces are obtained when the surfaces are situated in certain standard positions relative to the coordinate axes.

6.1 Ellipsoids

The *ellipsoid* is the quadric surface given by the equation

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*. \tag{6.1}$$

- The coordinate planes are all planes of symmetry of $\mathcal{E}$ since, for an arbitrary point $M(x, y, z) \in \mathcal{E}$, its symmetrical points with respect to these planes, $M_1(-x, y, z)$, $M_2(x, -y, z)$ and $M_3(x, y, -z)$ belong to $\mathcal{E}$; therefore, the coordinate axes are axes of symmetry for $\mathcal{E}$ and the origin O is the center of symmetry of the ellipsoid (6.1);

- The traces in the coordinates planes are ellipses of equations

$$\begin{cases} \frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \\ x=0 \end{cases} ; \quad \begin{cases} \frac{x^2}{a^2}+\frac{z^2}{c^2}-1=0 \\ y=0 \end{cases} ; \quad \begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \\ z=0 \end{cases} ;$$

- The sections with planes parallel to xOy are given by setting $z=\lambda$ in (6.1). Then, a section is of equations $\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{\lambda^2}{c^2} \\ z=\lambda \end{cases}$.

  – If $|\lambda|<c$, the section is an ellipse

$$\begin{cases} \frac{x^2}{\left(a\sqrt{1-\frac{\lambda^2}{c^2}}\right)^2}+\frac{y^2}{\left(b\sqrt{1-\frac{\lambda^2}{c^2}}\right)^2}=1 \\ z=\lambda \end{cases} ;$$

  – If $|\lambda|=c$, the intersection is reduced to one (tangency) point $(0,0,\lambda)$;

  – If $|\lambda|>c$, the plane $z=\lambda$ does not intersect the ellipsoid $\mathcal{E}$.

  The sections with planes parallel to xOz or yOz are obtained in a similar way.

The ellipsoid is presented in Figure 6.1.

Figure 6.1:

Particular case: If $a = b = c = R$, one obtains the sphere

$$x^2 + y^2 + z^2 - R^2 = 0$$

of center O and radius R.

6.2 Hyperboloids of One Sheet

The surface of equation

$$\mathcal{H}_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{6.2}$$

is called *hyperboloid of one sheet.*

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$; hence, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;
- The intersections with the coordinates planes are, respectively, of equations

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ x = 0 \\ \text{a hyperbola} \end{cases}; \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{a hyperbola} \end{cases}; \quad \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0 \\ z = 0 \\ \text{an ellipse} \end{cases};$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases}; \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = 1 - \dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases}; \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{cases};$$

The hyperboloid $\mathcal{H}_1$ is presented in Figure 6.2.

Remark: The surface $\mathcal{H}_1$ contains two families of lines. One has

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} \Leftrightarrow \left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right).$$

Figure 6.2:

The equations of the two families of lines are

$$d_\lambda : \begin{cases} \lambda\left(\dfrac{x}{a}+\dfrac{z}{c}\right) = 1+\dfrac{y}{b} \\ \dfrac{x}{a}-\dfrac{z}{c} = \lambda\left(1-\dfrac{y}{b}\right) \end{cases}, \lambda \in \mathbb{R} \text{ and } d'_\mu : \begin{cases} \mu\left(\dfrac{x}{a}+\dfrac{z}{c}\right) = 1-\dfrac{y}{b} \\ \dfrac{x}{a}-\dfrac{z}{c} = \mu\left(1+\dfrac{y}{b}\right) \end{cases}, \mu \in \mathbb{R}.$$

Through any point on $\mathcal{H}_1$ pass two lines, one line from each family.

6.3 Hyperboloids of Two Sheets

The *hyperboloid of two sheets* is the surface of equation

$$\mathcal{H}_2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0, \qquad a,b,c \in \mathbb{R}_+^*. \tag{6.3}$$

- The coordinate planes are planes of symmetry for $\mathcal{H}_1$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{H}_1$;

- The intersections with the coordinates planes are, respectively,

$$\begin{cases} \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}+1=0 \\ x=0 \\ \text{a hyperbola;} \end{cases} ; \quad \begin{cases} \dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}+1=0 \\ y=0 \\ \text{a hyperbola} \end{cases} ; \quad \begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+1=0 \\ z=0 \\ \text{the empty set} \end{cases} ;$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1 - \dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases}, \qquad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -1 - \dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases}$$

and $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 + \dfrac{\lambda^2}{c^2} \\ z = \lambda \end{cases}$.

  - If $|\lambda| > c$, the section is an ellipse;
  - If $|\lambda| = c$, the intersection reduces to a point $(0, 0, \lambda)$;
  - If $|\lambda| < c$, one obtains the empty set.

The hyperboloid of two sheets is presented in Figure 6.3.

Figure 6.3:

6.4 Elliptic Cones

The surface of equation

$$\mathcal{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \qquad a, b, c \in \mathbb{R}_+^*, \tag{6.4}$$

is called *elliptic cone*.

- The coordinate planes are planes of symmetry for $\mathcal{C}$, the coordinate axes are axes of symmetry and the origin O is the center of symmetry of $\mathcal{C}$;
- The intersections with the coordinates planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0 \\ x = 0 \\ \text{two lines} \end{cases}; \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} - 1 = 0 \\ y = 0 \\ \text{two lines} \end{cases}; \quad \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \\ z = 0 \\ \text{the origin } O(0,0,0) \end{cases};$$

- The intersections with planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{a^2} \\ x = \lambda \\ \text{hyperbolas} \end{cases}; \quad \begin{cases} \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2} = -\dfrac{\lambda^2}{b^2} \\ y = \lambda \\ \text{hyperbolas} \end{cases}; \quad \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{\lambda^2}{c^2} \\ z = \lambda \\ \text{ellipses} \end{cases};$$

The cone is presented in Figure 6.4.

Figure 6.4:

Remark: If $a = b$ in (6.4), one obtains the equation of a *circular cone*.

6.5 Elliptic Paraboloids

The surface of equation

$$\mathcal{P}_e : \frac{x^2}{p} + \frac{y^2}{q} = 2z, \qquad p, q \in \mathbb{R}_+^*, \tag{6.5}$$

is called *elliptic paraboloid*.

- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are

$$\begin{cases} \dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{cases} ; \quad \begin{cases} \dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{cases} ; \quad \begin{cases} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{the origin } O(0,0,0) \end{cases} ;$$

- The intersection with the planes parallel to the coordinate planes are $\begin{cases} \dfrac{x^2}{p} + \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \end{cases}$,
  - If $\lambda > 0$, the section is an ellipse;
  - If $\lambda = 0$, the intersection reduces to the origin;
  - If $\lambda < 0$, one has the empty set;

  and

$$\begin{cases} \dfrac{y^2}{q} = 2z - \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases} ; \quad \begin{cases} \dfrac{x^2}{p} = 2z - \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas} \end{cases} ;$$

The elliptic paraboloid is presented in Figure 6.5.

Figure 6.5:

6.6 Hyperbolic Paraboloids

The *hyperbolic paraboloid* is the surface given by the equation

$$\mathcal{P}_h : -\frac{x^2}{p} + \frac{y^2}{q} = 2z, \qquad p, q \in \mathbb{R}_+^*. \tag{6.6}$$

- The planes xOz and yOz are planes of symmetry;
- The traces in the coordinate planes are, respectively,

$$\begin{cases} \dfrac{y^2}{q} = 2z \\ x = 0 \\ \text{a parabola} \end{cases}; \quad \begin{cases} -\dfrac{x^2}{p} = 2z \\ y = 0 \\ \text{a parabola} \end{cases}; \quad \begin{cases} -\dfrac{x^2}{p} + \dfrac{y^2}{q} = 0 \\ z = 0 \\ \text{two lines} \end{cases};$$

- The intersection with the planes parallel to the coordinate planes are

$$\begin{cases} \dfrac{y^2}{q} = 2z + \dfrac{\lambda^2}{p} \\ x = \lambda \\ \text{parabolas} \end{cases}; \quad \begin{cases} -\dfrac{x^2}{p} = 2z - \dfrac{\lambda^2}{q} \\ y = \lambda \\ \text{parabolas} \end{cases}; \quad \begin{cases} -\dfrac{x^2}{p} + \dfrac{y^2}{q} = 2\lambda \\ z = \lambda \\ \text{hyperbolas} \end{cases};$$

The elliptic paraboloid is presented in Figure 6.6.

Figure 6.6:

Remark: The hyperbolic paraboloid contains two families of lines. Since

$$\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right)\left(-\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z,$$

then the two families are, respectively, of equations

$$d_\lambda : \begin{cases} -\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = \lambda \\ \lambda\left(\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}}\right) = 2z \end{cases}, \lambda \in \mathbb{R}^* \text{ and } d'_\mu : \begin{cases} \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = \mu \\ \mu\left(-\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}}\right) = 2z \end{cases}, \mu \in \mathbb{R}^*.$$

6.7 Singular Quadrics

6.7.1 Elliptic Cylinder

The *elliptic cylinder* is the surface of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0, \quad (\text{or } \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0, \text{ or } \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0). \quad (6.7)$$

The graph of the elliptic cylinder is presented in Figure 6.7.

Figure 6.7:

6.7.2 Hyperbolic Cylinder

The *hyperbolic cylinder* is the surface of equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \quad a, b > 0, \quad (\text{or } \frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0, \text{ or } \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0). \quad (6.8)$$

The graph of the hyperbolic cylinder is presented in Figure 6.8.

Figure 6.8:

6.7.3 Parabolic Cylinder

The *parabolic cylinder* is the surface of equation

$$y^2 = 2px, \quad p > 0, \quad \text{(or an alternative equation).} \tag{6.9}$$

The graph of the parabolic cylinder is presented in Figure 6.9.

Figure 6.9:

6.7.4 A Pair of Two Planes With Nonempty Intersection

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad a, b > 0,$$

(or an analogous form) represents the union of two planes $\pi_1 : \dfrac{x}{a} - \dfrac{y}{b} = 0$ and $\pi_2 : \dfrac{x}{a} + \dfrac{y}{b} = 0$.

6.7.5 A Pair of Two Parallel Planes

The equation

$$x^2 - a^2 = 0, \quad a > 0,$$

(or a similar form) represents the union of two parallel planes $\pi_1 : x - a = 0$ and $\pi_2 : x + a = 0$.

6.7.6 Two Identical Planes

The equation $x^2 = 0$, for instance, represents the union of two identical planes.

6.7.7 The Line

The equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0$ represents a line in 3-space (in fact, the Oz axes).

6.7.8 The Point

The point $P_0(x_0, y_0, z_0)$ is given by the solution of the equation

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 0.$$

6.7.9 The Empty Set

It can be seen as the quadric given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$$

6.8 The Classification Algorithm

Let Q be the quadric surface given by the zeros of the polynomial

$$g(x,y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{10}x + 2a_{20}y + 2a_{30}x + a_{00}$$

with $a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{12}^2 + a_{23}^2 + a_{13}^2 > 0$.

Consider the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{12} & a_{22} & a_{23} \\ a_{30} & a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Denote by r =rank A, r' =rank A', $\delta = \det A$ and $\Delta = \det A'$. Let $P(\lambda)$ be the characteristic polynomial of the matrix A, i.e.

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{12} & a_{22} - \lambda & a_{23} \\ a_{13} & a_{23} & a_{33} - \lambda \end{vmatrix}.$$

Since A is symmetric, then $P(\lambda)$ has three real roots λ_1, λ_2 and λ_3. These are the eigenvalues of A. Let i be the number of negative eigenvalues (obviously, $0 \leq i \leq 3$).

Now, the characteristic polynomial is

$$P(\lambda) = I_0 - I_1\lambda + I_2\lambda^2 - \lambda^3,$$

where

$$I_0 = \delta = \lambda_1\lambda_2\lambda_3,$$

$$I_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$

$$I_2 = a_{11} + a_{22} + a_{33} = \mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \lambda_3.$$

Consider

$$P_0(\lambda) = \begin{vmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{10} & a_{11} - \lambda & a_{12} & a_{13} \\ a_{20} & a_{12} & a_{22} - \lambda & a_{23} \\ a_{30} & a_{13} & a_{23} & a_{33} - \lambda \end{vmatrix} = k_0 - k_1\lambda + k_2\lambda^2 - \lambda^3,$$

where

$$k_0 = \Delta,$$

$$k_1 = \begin{vmatrix} a_{00} & a_{10} & a_{20} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{10} & a_{30} \\ a_{10} & a_{11} & a_{13} \\ a_{30} & a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{20} & a_{30} \\ a_{20} & a_{22} & a_{23} \\ a_{30} & a_{23} & a_{33} \end{vmatrix},$$

$$k_2 = \begin{vmatrix} a_{00} & a_{30} \\ a_{30} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{20} \\ a_{20} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{10} \\ a_{10} & a_{11} \end{vmatrix}.$$

Theorem 6.8.1. *Let Q be the quadric surface given by $g(x, y, z) = 0$, where g is a polynomial of degree 2. There exists a Cartesian system of coordinates in which the quadric has one of the following forms:*

1) If $\delta \neq 0$, then $Q : \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + \dfrac{\Delta}{\delta} = 0$;

2) If $\delta = 0$, $\Delta \neq 0$, $I_1 \neq 0$, then $Q : \lambda_1 x'^2 + \lambda_2 y'^2 \pm \sqrt{-\dfrac{\Delta}{I_1}} z' = 0$;

3) If $\delta = 0$, $\Delta = 0$, $I_1 \neq 0$, then $Q : \lambda_1 x'^2 + \lambda_2 y'^2 + \dfrac{k_1}{I_1} = 0$;

4) If $\delta = 0$, $I_1 = 0$, $k_1 \neq 0$, then $Q : I_2 x'^2 + 2\sqrt{-\dfrac{k_1}{I_2}} y' = 0$;

5) If $\delta = 0$, $I_1 = 0$, $k_1 = 0$, then $Q : I_2 x'^2 + \dfrac{k_2}{I_2} = 0$.

Conclusions

<table>
<tr><th>r'</th><th>r</th><th>i</th><th colspan="2">Quadric</th><th>Canonical equation</th><th>Remarks</th></tr>
<tr><td rowspan="6">4</td><td rowspan="4">3</td><td rowspan="2">3</td><td>$\Delta > 0$</td><td>$\emptyset$</td><td rowspan="4">$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + \frac{\Delta}{\delta} = 0$</td><td rowspan="6">non-singular quadrics</td></tr>
<tr><td>$\Delta < 0$</td><td>ellipsoid</td></tr>
<tr><td rowspan="2">2</td><td>$\Delta > 0$</td><td>hyperboloid of one sheet</td></tr>
<tr><td>$\Delta < 0$</td><td>hyperboloid of two sheets</td></tr>
<tr><td rowspan="2">2</td><td>2</td><td colspan="2">elliptic paraboloid</td><td rowspan="2">$\lambda_1 x'^2 + \lambda_2 y'^2 \pm \sqrt{-\frac{\Delta}{I_1}} z' = 0$</td></tr>
<tr><td>1</td><td colspan="2">hyperbolic paraboloid</td></tr>
<tr><td rowspan="6">3</td><td rowspan="2">3</td><td>3</td><td colspan="2">a point</td><td rowspan="2">$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 = 0$</td><td rowspan="11">singular quadrics</td></tr>
<tr><td>2</td><td colspan="2">elliptic cone</td></tr>
<tr><td rowspan="3">2</td><td rowspan="2">2</td><td>$k_1 I_1 < 0$</td><td>elliptic cylinder</td><td rowspan="3">$\lambda_1 x'^2 + \lambda_2 y'^2 + \frac{k_1}{I_1} = 0$</td></tr>
<tr><td>$k_1 I_1 > 0$</td><td>$\emptyset$</td></tr>
<tr><td>1</td><td colspan="2">hyperbolic cylinder</td></tr>
<tr><td>1</td><td>1</td><td colspan="2">parabolic cylinder</td><td>$I_2 x'^2 + 2\sqrt{-\frac{k_1}{I_2}} y' = 0$</td></tr>
<tr><td rowspan="4">2</td><td rowspan="2">2</td><td>2</td><td colspan="2">a line</td><td rowspan="2">$\lambda_1 x'^2 + \lambda_2 y'^2$</td></tr>
<tr><td>1</td><td colspan="2">two planes</td></tr>
<tr><td rowspan="2">1</td><td rowspan="2">1</td><td>$k_2 > 0$</td><td>$\emptyset$</td><td rowspan="2">$I_2 x'^2 + \frac{k_2}{I_2} = 0$</td></tr>
<tr><td>$k_2 < 0$</td><td>two parallel planes</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td colspan="2">a double plane</td><td>$x^2 = 0$</td></tr>
</table>

6.9 Exercises

1) Sketch the graph of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$.

2) Sketch the graph of the hyperboloid of one sheet $x^2 + y^2 - \frac{z^2}{4} = 1$.

3) Sketch the graph of the hyperboloid of two sheets $x^2 + \frac{y^2}{4} - z^2 = -1$.

4) Sketch the graph of the elliptic cone $z^2 = x^2 + \dfrac{y^2}{4}$.

5) Sketch the graph of the elliptic paraboloid $z = \dfrac{x^2}{4} + \dfrac{y^2}{9}$.

6) Sketch the graph of the hyperbolic paraboloid $z = \dfrac{x^2}{4} - \dfrac{y^2}{9}$.

7) Sketch the graph of the surface $z = 1 - x^2 - y^2$.

8) Sketch the graph of the surface $4x^2 + 4y^2 + z^2 + 8y - 4z = -4$.

9) Show that the lines $\begin{cases} x = 3 + t \\ y = 2 + t \\ z = 5 + 2t \end{cases}$ and $\begin{cases} x = 3 + t \\ y = 2 - t \\ z = 5 + 10t \end{cases}$ are completely contained into the hyperbolic paraboloid $z = x^2 - y^2$.

10) Name and sketch the surface

a) $z = (x+2)^2 + (y-3)^2 - 9$;
b) $4x^2 - y^2 + 16(z-2)^2 = 10$;
c) $4x^2 + y^2 - z^2 + 8x - 2y + 4z = 0$;
d) $9x^2 + y^2 + 4z^2 - 18x + 2y + 16z - 10 = 0$.

Chapter 7

Generated Surfaces

Consider the 3-dimensional Euclidean space $\mathcal{E}_3$, together with a Cartesian system of coordinates $Oxyz$. Generally, the set

$$S = \{M(x, y, z) : F(x, y, z) = 0\},$$

where $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a real function and D is a domain, is called *surface* of implicit equation $F(x, y, z) = 0$ (the quadric surfaces, defined in the previous chapter for F a polynomial of degree two, are such of surfaces). On the other hand, the set

$$S_1 = \{M(x, y, z) : x = x(u, v), y = y(u, v), z = z(u, v)\},$$

where $x, y, z : D_1 \subseteq \mathbb{R}^2 \to \mathbb{R}$, is a *parameterized surface*, of parametric equations

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \quad (u, v) \in D_1.$$

The intersection between two surfaces is a *curve* in 3-space (remember, for instance, that the intersection between a quadric surface and a plane is a conic section, hence the conics are plane curves). Then, the set

$$C = \{M(x, y, z) : F(x, y, z) = 0, G(x, y, z) = 0\},$$

where $F, G : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, is the curve of *implicit* equations

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}.$$

As before, one can parameterize the curve. The set

$$C_1 = \{M(x, y, z) : x = x(t), y = y(t), z = z(t)\},$$

where $x, y, z : I \subseteq \mathbb{R} \to \mathbb{R}$ and I is open, is called *parameterized curve* of parametric equations

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in I.$$

Let be given a family of curves, depending on one single parameter λ,

$$\mathcal{C}_\lambda : \begin{cases} F_1(x, y, z; \lambda) = 0 \\ F_2(x, y, z; \lambda) = 0 \end{cases}.$$

In general, the family $\mathcal{C}_\lambda$ does not cover the entire space. By eliminating the parameter λ between the two equations of the family, one obtains the equation of the surface *generated* by the family of curves.

Suppose now that the family of curves depends on two parameters λ, μ,

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} F_1(x, y, z; \lambda, \mu) = 0 \\ F_2(x, y, z; \lambda, \mu) = 0 \end{cases},$$

and that the parameters are related through $\varphi(\lambda, \mu) = 0$ (one can choose only the sub-family corresponding to such λ and μ). If it can be obtained an equation which does not depend on the parameters (by eliminating the parameters between the three equations), then the set of all the points which verify it is called surface *generated* by the family (or the sub-family) of curves.

7.1 Cylindrical Surfaces

The surface generated by a variable line (the *generatrix*), which remains parallel to a fixed line d and intersects a given curve $\mathcal{C}$, is called *cylindrical surface.* The curve $\mathcal{C}$ is called the *director curve* of the cylindrical surface.

Theorem 7.1.1. *The cylindrical surface, with the generatrix parallel to the line d, where*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases},$$

and having the director curve $\mathcal{C}$, where

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

Figure 7.1:

(suppose that d and $\mathcal{C}$ are not coplanar), is characterized by an equation of the form

$$\varphi(\pi_1, \pi_2) = 0. \tag{7.1}$$

Proof: An arbitrary line, which is parallel to $d : \begin{cases} \pi_1(x, y, z) = 0 \\ \pi_2(x, y, z) = 0 \end{cases}$, will be of equations

$$d_{\lambda,\mu} : \begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}.$$

Of course, not every line from the family $d_{\lambda,\mu}$ intersects the curve $\mathcal{C}$. This happens only when the system of equations

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \end{cases}$$

is compatible. By eliminating x, y and z between the four equations of the system, one obtains a *compatibility condition* for the parameters λ and μ,

$$\varphi(\lambda, \mu) = 0.$$

The equation of the surface can be determined now from the system

$$\begin{cases} \pi_1(x, y, z) = \lambda \\ \pi_2(x, y, z) = \mu \\ \varphi(\lambda, \mu) = 0 \end{cases},$$

and it is immediate that $\varphi(\pi_1, \pi_2) = 0$. $\square$

Remark: Any equation of the form (7.1), where π_1 and π_2 are linear function of x, y and z, represents a cylindrical surface, having the generatrices parallel to $d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$.

Example: Let us find the equation of the cylindrical surface having the generatrices parallel to

$$d : \begin{cases} x + y = 0 \\ z = 0 \end{cases}$$

and the director curve given by

$$\mathcal{C} : \begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \end{cases}.$$

The equations of the generatrices d are

$$d_{\lambda,\mu} : \begin{cases} x + y = \lambda \\ z = \mu \end{cases}.$$

They must intersect the curve $\mathcal{C}$, i.e. the system

$$\begin{cases} x^2 - 2y^2 - z = 0 \\ x - 1 = 0 \\ x + y = \lambda \\ z = \mu \end{cases}$$

has to be compatible. A solution of the system can be obtained using the three last equations

$$\begin{cases} x = 1 \\ y = \lambda - 1 \\ z = \mu \end{cases}$$

and, replacing in the first one, one obtains the compatibility condition

$$2(\lambda - 1)^2 + \mu - 1 = 0.$$

The equation of the surface is obtained by eliminating the parameters in

$$\begin{cases} x + y = \lambda \\ z = \mu \\ 2(\lambda - 1)^2 + \mu - 1 = 0 \end{cases}.$$

Then,

$$2(x + y - 1)^2 + x - 1 = 0.$$

7.2 Conical Surfaces

The surface generated by a variable line, which passes through a fixed point V and intersects a given curve $\mathcal{C}$, is called *conical surface.* The point V is called the *vertex* of the surface and the curve $\mathcal{C}$ *director curve.*

Figure 7.2:

Theorem 7.2.1. *The conical surface, of vertex* $V(x_0, y_0, z_0)$ *and director curve*

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases},$$

(suppose that V *and* $\mathcal{C}$ *are not coplanar), is characterized by an equation of the form*

$$\varphi\left(\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}\right) = 0. \tag{7.2}$$

Proof: The equations of an arbitrary line through $V(x_0, y_0, z_0)$ are

$$d_{\lambda\mu} : \begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \end{cases}.$$

A generatrix has to intersect the curve $\mathcal{C}$, hence the system of equations

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

must be compatible. This happens for some values of the parameters λ and μ, which verify a *compatibility condition*

$$\varphi(\lambda, \mu),$$

obtained by eliminating x, y and z in the the previous system of equations.

In these conditions, the surface is generated and its equation rises from the system

$$\begin{cases} x - x_0 = \lambda(z - z_0) \\ y - y_0 = \mu(z - z_0) \\ \varphi(\lambda, \mu) = 0 \end{cases}.$$

It follows that

$$\varphi\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0.. \quad \square$$

Remark: If φ is an algebraic function, then the equation (7.2) can be written in the form $\phi(x - x_0, y - y_0, z - z_0) = 0$, where ϕ is homogeneous with respect to $x - x_0$, $y - y_0$ and $z - z_0$.

If φ is algebraic and V is the origin of the system of coordinates, then the equation of the conical surface is $\phi(x, y, z) = 0$, with ϕ a homogeneous polynomial. Conversely, an algebraic homogeneous equation in x, y and z represents a conical surface with the vertex at the origin.

Example: Let us determine the equation of the conical surface, having the vertex $V(1, 1, 1)$ and the director curve

$$\mathcal{C} : \begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \end{cases}.$$

The family of lines passing through V has the equations

$$d_{\lambda\mu} : \begin{cases} x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}.$$

The system of equations

$$\begin{cases} (x^2 + y^2)^2 - xy = 0 \\ z = 0 \\ x - 1 = \lambda(z - 1) \\ y - 1 = \mu(z - 1) \end{cases}$$

must be compatible. A solution is

$$\begin{cases} x = 1 - \lambda \\ y = 1 - \mu \\ z = 0 \end{cases},$$

and, replaced in the first equation of the system, gives the compatibility condition

$$[(1-\lambda)^2 + (1-\mu)^2]^2 - (1-\lambda)(1-\mu) = 0.$$

The equation of the conical surface is obtained by eliminating the parameters λ and μ in

$$\begin{cases} x - 1 = \lambda(z-1) \\ y - 1 = \mu(z-1) \\ ((1-\lambda)^2 + (1-\mu)^2)^2 - (1-\lambda)(1-\mu) = 0 \end{cases}.$$

Expressing $\lambda = \dfrac{x-1}{z-1}$ and $\mu = \dfrac{y-1}{z-1}$ and replacing in the compatibility condition, one obtains

$$\left[\left(\frac{z-x}{z-1}\right)^2 + \left(\frac{z-y}{z-1}\right)^2\right]^2 - \left(\frac{z-x}{z-1}\right)\left(\frac{z-y}{z-1}\right) = 0,$$

or

$$[(z-x)^2 + (z-y)^2]^2 - (z-x)(z-y)(z-1)^2 = 0.$$

7.3 Conoidal Surfaces

The surface generated by a variable line, which intersects a given line d and a given curve $\mathcal{C}$, and remains parallel to a given plane π, is called *conoidal surface.* The curve $\mathcal{C}$ is the *director curve* and the plane π is the *director plane* of the conoidal surface.

Theorem 7.3.1. *The conoidal surface whose generatrix intersects the line*

$$d : \begin{cases} \pi_1 = 0 \\ \pi_2 = 0 \end{cases}$$

and the curve

$$\mathcal{C} : \begin{cases} F_1(x,y,z) = 0 \\ F_2(x,y,z) = 0 \end{cases}$$

Figure 7.3:

and has the director plane $\pi = 0$, (suppose that π is not parallel to d and that $\mathcal{C}$ is not contained into π), is characterized by an equation of the form

$$\varphi\left(\pi, \frac{\pi_1}{\pi_2}\right) = 0. \tag{7.3}$$

Proof: An arbitrary generatrix of the conoidal surface is contained into a plane parallel to π and, on the other hand, comes from the bundle of planes containing d. Then, the equations of a generatrix are

$$d_{\lambda\mu} : \begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \end{cases} .$$

Again, the generatrix must intersect the director curve, hence the system of equations

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases}$$

has to be compatible. This leads to a compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

and the equation of the conoidal surface is obtained from

$$\begin{cases} \pi = \lambda \\ \pi_1 = \mu\pi_2 \\ \varphi(\lambda, \mu) = 0 \end{cases} .$$

By expressing λ and μ, one obtains (7.3). $\square$

Example: Let us find the equation of the conoidal surface, whose generatrices are parallel to xOy and intersect Oz and the curve

$$\begin{cases} y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases}.$$

The equations of xOy and Oz are, respectively,

$$xOy : z = 0, \quad \text{and} \quad Oz : \begin{cases} x = 0 \\ z = 0 \end{cases},$$

so that the equations of the generatrix are

$$d_{\lambda,\mu} : \begin{cases} x = \lambda y \\ z = \mu \end{cases}.$$

From the compatibility of the system of equations

$$\begin{cases} x = \lambda y \\ z = \mu \\ y^2 - 2z + 2 = 0 \\ x^2 - 2z + 1 = 0 \end{cases},$$

one obtains the compatibility condition

$$2\lambda^2\mu - 2\lambda^2 - 2\mu + 1 = 0,$$

and, replacing $\lambda = \dfrac{y}{x}$ and $\mu = z$, the equation of the conoidal surface is

$$2x^2z - 2y^2z - 2x^2 + y^2 = 0.$$

7.4 Revolution Surfaces

The surface generated after the rotation of a given curve $\mathcal{C}$ around a given line d is said to be a *revolution surface*.

Theorem 7.4.1. *The equation of the surface generated by the curve*

$$\mathcal{C} : \begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{cases},$$

Figure 7.4:

in its rotation around the line

$$d : \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{r},$$

is of the form

$$\varphi((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2, px + qy + rz) = 0. \tag{7.4}$$

Proof: An arbitrary point on the curve $\mathcal{C}$ will describe, in its rotation around d, a circle situated into a plane orthogonal on d and having the center on the line d. This circle can be seen as the intersection between a sphere, having the center on d and of variable radius, and a plane, orthogonal on d, so that its equations are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}.$$

The circle has to intersect the curve $\mathcal{C}$, therefore the system

$$\begin{cases} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \lambda \\ px + qy + rz = \mu \end{cases}$$

must be compatible. One obtains the compatibility condition

$$\varphi(\lambda, \mu) = 0,$$

which, after replacing the parameters, gives the equation of the surface (7.4).
$\square$

Example: Let us determine the equation of the *torus* (the surface generated by a circle $\mathcal{C}$, which turns around an exterior line, lying in the plane of the circle). Choose the system of coordinates such that Oz is the line d and Ox is the orthogonal line on d, passing through the center of $\mathcal{C}$. Let r be the radius of the circle and $(0, a, 0)$ the coordinates of its center. Since the line is exterior to the circle, then $a > r > 0$.

Figure 7.5:

In this system of coordinates, the equations of the circle and of the line are, respectively,

$$\mathcal{C} : \begin{cases} (y-a)^2 + z^2 = r^2 \\ x = 0 \end{cases} \quad \text{and} \quad d : \begin{cases} x = 0 \\ z = 0 \end{cases} .$$

The equations of the family of circles generating the surface are

$$\mathcal{C}_{\lambda,\mu} : \begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \end{cases} .$$

The system of equations

$$\begin{cases} x^2 + y^2 + z^2 = \lambda \\ z = \mu \\ x = 0 \\ (y-a)^2 + z^2 = r^2 \end{cases}$$

must be compatible. Choose the first three equations in order to obtain a solution of the system

$$\begin{cases} x = 0 \\ y = \pm\sqrt{\lambda - \mu^2} \\ z = \mu \end{cases} .$$

Replacing in the remained equation, one obtains the compatibility condition

$$(\pm\sqrt{\lambda-\mu^2}-a)^2+\mu^2=r^2.$$

The equation of the torus is

$$(\pm\sqrt{x^2+y^2}-a)^2+z^2=r^2,$$

or

$$(x^2+y^2+z^2+a^2-r^2)^2=4a^2(x^2+y^2).$$

7.5 Exercises

1) Find the equation of the cylindrical surface, having the circle

$$\mathcal{C}:\begin{cases} x^2+y^2-a^2 \\ z=0 \end{cases}$$

as director curve, and the generatrices parallel to $d: x=y=z$.

2) Find the equation of the cylindrical surface, generated by a variable line of direction $(1,2,-1)$ and of director curve

$$\mathcal{C}:\begin{cases} x+y+z=0 \\ x^2+y^2+z^2=4 \end{cases}.$$

3) Find the equation of the cylindrical surface, having the director curve

$$\mathcal{C}:\begin{cases} x^2+y^2-z=0 \\ x=2z \end{cases}$$

and the generatrices orthogonal on the plane containing $\mathcal{C}$.

4) Find the equation of the conical surface, with the vertex $V(0,-a,0)$ and the director curve

$$\mathcal{C}:\begin{cases} x^2+y^2+z^2=4 \\ y+z=2 \end{cases}.$$

5) A circular disk of center $(1,0,2)$ and radius 1 is parallel to the plane yOz. Supposing that there is a light source at the point $P(0,0,3)$, find the shadow that the disk darts on the plane xOy.

6) Find the geometric locus of the lines passing through the origin $O(0,0,0)$ and tangent to the sphere

$$(x-5)^2+(y+1)^2+z^2-16=0.$$

7) Find the equation of the surface generated by a line which intersects Oz, the line

$$d:\begin{cases} x-z=0 \\ x+2y-3=0 \end{cases}$$

and stays parallel to xOy.

8) Find the equation of the conoidal surface, generated by a line which intersects a line d, a circle situated in a plane parallel to d and which stays parallel to a plane orthogonal on d (the Willis' conoid).

9) Find the equation of the surface generated by the rotation of the line $d: x=y=z$ around the axis Oz.

10) Find the equation of the surface generated by the rotation of a line around another line.

11) Find the equation of the surface generated by the rotation of the hyperbola

$$\mathcal{H}:\begin{cases} z=0 \\ \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-1=0 \end{cases}$$

around the axis Oy.